# $p$-adic vector bundles on curves and abelian varieties and representations of the fundamental group

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# Contents

# 1 Deutschsprachige Zusammenfassung

## $p$-adische Vektorbündel auf Kurven und Abelschen Varietäten, und Darstellungen der Fundamentalgruppe

In vorliegender Arbeit werden verschiedene Zugänge zur $p$-adischen Integration und $p$-adischen Riemann-Hilbert-Korrespondenz untersucht und miteinander verglichen.

Für einen glatten $K$-analytischen Raum über einem abgeschlossenen Teilkörper $K$ von $\mathbb{C}_p$ hat V. Berkovich eine Theorie der $p$-adischen Integration und des Paralleltransports entlang Wegen entwickelt. Insbesondere kann man (lokal unipotenten) Vektorbündeln mit Zusammenhang diskrete Darstellungen der topologischen Fundamentalgruppe auf endlich dimensionale $K$-Vektorräume zuordnen.

In einer Arbeit von C. Deninger und A. Werner wird für eine Kategorie von Vektorbündeln $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$ auf einer glatten projektiven Kurve $X$ über $\overline{\mathbb{Q}_p}$ ein Paralleltransport entlang étaler Wegen definiert. Insbesondere kann man jedem dieser Vektorbündel eine stetige Darstellung auf einen endlich dimensionalen $\mathbb{C}_p$-Vektorraum der algebraischen Fundamentalgruppe von $X$ zuordnen, welche wir DeWe-Darstellungen nennen wollen. Zur gleichen Zeit wurde von G. Faltings eine $p$-adische Simpson-Korrespondenz beschrieben. Hierbei kann gewissen Vektorbündeln auf einer Kurve $X$, die mit einem Higgs-Feld ausgestattet sind, eine Darstellung der algebraischen Fundamentalgruppe von $X$ zugeordnet werden.

In der vorliegenden Arbeit wird gezeigt, dass man sogenannten temperierten Darstellungen der algebraischen Fundamentalgruppe von $X$ ein Vektorbündel mit kanonischem Zusammenhang zuordnen kann. Diese Vektorbündel liegen dann in der Kategorie $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$ und die zugehörigen Darstellungen sind mit den DeWe-Darstellungen kompatibel. Wir benutzen dann eine Methode von G. Herz, um die stetigen DeWe-Darstellungen mit den diskreten Darstellungen, die im Berkovich-Paralleltransport auftreten, zu vergleichen. Wir zeigen weiter, dass die Konstruktion von G. Faltings in diesem Fall mit den DeWe-Darstellungen übereinstimmt, falls das Higgs-Feld gleich Null ist. Wir haben folglich gezeigt, dass die obig genannten Zugänge zur $p$-adischen Integration und $p$-adischen Riemann-Hilbert Korrespondenz im Spezialfall der temperierten Darstellungen miteinander kompatibel sind.

In einer weiteren Arbeit von C. Deninger und A. Werner wurde die Theorie der DeWe-Darstellung auch für Abelsche Varietäten $A/\overline{\mathbb{Q}_p}$ mit guter Reduktion entwickelt. Wir konnten zeigen, dass die Kategorie der Vektorbündel

$\mathfrak{B}_{A_{\mathbb{C}_p}}$ auf einer Abelschen Varietät $A$, für die DeWe-Darstellungen definiert sind, genau aus den translations-invarianten Vektorbündeln besteht (unter der Annahme, dass der DeWe-Funktor volltreu ist). Im Falle gewöhnlicher Reduktion konnte auch gezeigt werden, dass die zugehörigen Darstellungen genau die temperierten Darstellungen sind. Insbesondere erhält man in diesem Fall eine Kategorienäquivalenz zwischen temperierten Darstellungen und translations-invarianten Vektorbündeln auf $A$, welche dann auch einen kanonischen Zusammenhang besitzen.

## 2 Introduction

On a Riemann surface $X$ there is a well-known correspondence (Riemann-Hilbert) between complex representations of the fundamental group $\pi_1(X, x)$ and $C^\infty$ vector bundles with flat connections on $X$. In this correspondence a local system corresponding to a representation gives rise to a vector bundle with connection if tensored with the structure sheaf. Conversely, the horizontal sections of a $C^\infty$ vector bundle with flat connection define a local system. By a theorem of C. Simpson [Sim92] $C^\infty$ vector bundles with flat connection correspond to holomorphic vector bundles equipped with a so-called Higgs-field.

There are some analogues of this theory for varieties and vector bundles defined over a $p$-adic field:

There is an algebraic $p$-adic analogue: Let $X$ be a smooth proper curve over $\overline{\mathbb{Q}_p}$. In [DeWe05b] A.Werner and C. Deninger defined functorial isomorphisms of parallel transport along étale paths for a class of vector bundles on $X_{\mathbb{C}_p} = X \times_{\mathrm{Spec}\overline{\mathbb{Q}_p}} \mathrm{Spec}\mathbb{C}_p$. The category of such vector bundles is denoted by $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$ and contains all vector bundles of degree 0 that have strongly semistable reduction. In particular, all vector bundles in $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$ give rise to continuous representations of the algebraic fundamental group on finite dimensional $\mathbb{C}_p$-vector spaces. Their construction also works for abelian varieties $A$ with good reduction. To each vector bundle on $A$ that lies in a certain category $\mathfrak{B}_{A_{\mathbb{C}_p}}$ they can associate a continuous representation of the Tate-module $TA$ of $A$ on finite dimensional $\mathbb{C}_p$-vector spaces. We will call the representations attached to vector bundles in their theory DeWe-representations.

At the same time G. Faltings [Fal05] established a $p$-adic Simpson correspondence. He showed that there is a correspondence between vector bundles on $X$ equipped with a Higgs-field $\theta$ and so called "generalized representations" which contain the representations of the algebraic fundamental group of $X$ as a full subcategory. It is assumed that the construction of G. Faltings is compatible with the construction of C. Deninger and A. Werner if the Higgs-field $\theta$ is zero.

It is an interesting question to characterize the vector bundles that give rise to representations without referring to the reduction behavior. It is also an interesting question to characterize the representations that correspond to zero Higgs-fields. Furthermore, connections on the vector bundles and horizontal sections are missing.

There is a topological $p$-adic analogue: In [Ber07] V. Berkovich developed a theory of $p$-adic integration and parallel transport along paths in the framework of Berkovich spaces. For a smooth $K$-analytic space $Y$ there are isomorphisms of parallel transport along paths for all locally unipotent vector bundles with connection on this space. The parallel transport involves the horizontal sections of these connections. To obtain a full set of horizontal sections one has to work with a sheaf $\mathcal{S}_Y^\lambda$ on $Y$ that is an extension of the structure sheaf $\mathcal{O}_Y$. In particular, each locally unipotent vector bundle with connection gives rise to a representation of the topological fundamental group of $Y$. A special case of locally unipotent vector bundles are vector bundles that are attached to discrete representations of the topological fundamental group on finite dimensional $K$-vector spaces. These were considered in [And03]. If one omits connections then there is theorem of G. Faltings [Fal83] (see also [PuRe86]) that classifies an interesting class of vector bundles in the case that $Y$ is a Mumford curve. He shows that there is a bijective correspondence between semi-stable vector bundles of degree 0 on $Y$ and $\Phi$-bounded representation of the topological fundamental group of $Y$ ($\Phi$-bounded representation are representations that satisfy certain growth conditions).

In this thesis we want to find a relation between the algebraic and the analytic approach. We encounter several problems:

a) Representations in the topological case are of discrete nature, whereas algebraic representation are continuous with respect to the $p$-adic topology.

b) In the topological case vector bundles always have a connection, whereas in the algebraic setting connections do not always appear.

c) In Faltings' work one has to lift everything to the dual numbers of a Fontaine ring to get a correspondence that is independent of certain choices. The construction depends also on the choice of an exponential function for the multiplicative group.

d) The parallel transport of Berkovich involves a sheaf of locally analytic functions $\mathcal{S}_Y^\lambda$.

Problem a) was partly solved by G. Herz in his dissertation [Her05] in the case of Mumford curves. Problem b) is partly solved, because certain algebraic vector bundles have a canonical connection. We conjecture that every vector bundle lying in one of the categories $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$ or $\mathfrak{B}_{A_{\mathbb{C}_p}}$ defined by C. Deninger and A. Werner admits a canonical connection.

For Problem c) we omit the dual numbers and the Higgs field and use the work of [DeWe05b].

For Problem d) we only consider vector bundles that are attached to representations of the topological fundamental group. Then the parallel transport is more elementary, since the sheaf $\mathcal{S}_Y^\lambda$ is not needed anymore.

For an abelian variety $A$ over $\overline{\mathbb{Q}_p}$ having good reduction we show that the category $\mathfrak{B}_{A_{\mathbb{C}_p}}$ of vector bundles on $A_{\mathbb{C}_p}$ that give rise to DeWe-representations, is equal to the category of translation-invariant vector bundles on $A_{\mathbb{C}_p}$ (under the assumption, that the DeWe-functor is fully faithful). We also give a characterization of DeWe-representations that are attached to vector bundles in $\mathfrak{B}_{A_{\mathbb{C}_p}}$ if they are defined over a finite extension $K$ of $\mathbb{Q}_p$. In the case that $A$ has good ordinary reduction one obtains exactly the so called temperate representations of $TA$. This implies also that the vector bundles in $\mathfrak{B}_{A_{\mathbb{C}_p}}$ are equipped with a canonical integrable connection.

Using the theory of temperate representations we can generalize the comparison theorem of G. Herz [Her05] between the theory of DeWe-representations and Faltings $\Phi$-bounded representations on Tate-elliptic curves to non-integral coefficients.

As an application of canonical connections one can combine Faltings' $p$-adic Simpson correspondence with canonical connections to obtain a $p$-adic Riemann-Hilbert correspondence on curves. We could prove the existence of canonical connections for vector bundles on elliptic curves with ordinary (good or bad) reduction and for line bundles of degree 0 on curves with ordinary reduction.

We give a short outline of the paper: In Section 3 we recall various results that are already documented in the literature. In Section 4 we recall in detail how to attach a vector bundle with connection to a continuous representation of the étale fundamental group of a projective $\overline{\mathbb{Z}_p}$-model $\mathcal{X}$ of a projective variety $X$ defined over $\overline{\mathbb{Q}_p}$. This construction is motivated by the Katz correspondence between representations and $F$-crystals. Using descent theory this construction works also after a finite étale Galois covering $Y \to X$ of the generic fiber. In Section 5 we relate the topological parallel transport of V. Berkovich to the algebraic parallel transport of C. Deninger and A. Werner for some vector bundles with connections on curves. This builds on and generalizes the comparison theorem of G. Herz. In Section 6 we show that for an abelian variety over $\overline{\mathbb{Q}_p}$ with good reduction the category $\mathfrak{B}_{A_{\mathbb{C}_p}}$ defined by DeWe is equal to the category of homogeneous vector bundles on $A$. In Section 7 we give some applications of the previous results.

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# 3 Background

In this section we recall some known results about relations between $p$-adic vector bundles on varieties and representations of their fundamental groups (algebraic and analytic).

## 3.1 Notations

Let $p$ be a prime number. Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of the $p$-adic numbers $\mathbb{Q}_p$. The completion of $\overline{\mathbb{Q}_p}$ is denoted by $\mathbb{C}_p$, i.e. $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$. We write $\mathfrak{o}$ for the ring of integers in $\mathbb{C}_p$, and denote its reduction modulo $p^n$ by $\mathfrak{o}_n := \mathfrak{o}/p^n\mathfrak{o}$. In this case the ring $\mathfrak{o}_n$ is isomorphic to the reduction modulo $p^n$ of the $p$-adic integers $\overline{\mathbb{Z}_p}$ of $\overline{\mathbb{Q}_p}$, i.e. $\mathfrak{o}_n = \mathfrak{o}/p^n\mathfrak{o} \cong \overline{\mathbb{Z}_p}/p^n\overline{\mathbb{Z}_p}$. For a scheme, a sheaf, a morphism or a representation that is defined over $\mathfrak{o}$ or $\overline{\mathbb{Z}_p}$ a subscript $n$ will denote its reduction modulo $p^n$, e.g if $\mathcal{X} := \mathrm{Spec}\overline{\mathbb{Z}_p}[X]$, then $\mathcal{X}_n = \mathcal{X} \times_{\overline{\mathbb{Z}_p}} \overline{\mathbb{Z}_p}/p^n\overline{\mathbb{Z}_p}$ . If $\rho : \pi_1 \to Aut_{\mathfrak{o}}(\mathbb{L})$ is a representation of a group $\pi_1$ on a free $\mathfrak{o}$-module $\mathbb{L}$ of rank $r$, then $\rho_{\mathbb{C}_p}$ denotes its extension to $Aut_{\mathbb{C}_p}(\mathbb{L} \otimes_{\mathfrak{o}} \mathbb{C}_p)$. If $X/\mathrm{Spec}\overline{\mathbb{Q}_p}$ is a scheme, then $X_{\mathbb{C}_p}$ denotes its base change with $\mathbb{C}_p$. If $X$ is a variety over a complete non-archimedean field, then $X^{an}$ resp. $X^{rig}$ denotes its analytification (Berkovich), resp. its rigidification.

## 3.2 Deninger-Werner parallel transport on curves

$X$ a smooth projective curve over $\overline{\mathbb{Q}_p}$

We will review the étale parallel transport defined by Deninger and Werner in the case of curves [DeWe05b].

**Definition 3.1 (DeWe).** - Let $V$ be a valuation ring with quotient field $Q$.

a) A model of a smooth projective curve $C$ over $Q$ is a finitely presented, flat and proper scheme $\mathcal{C}$ over $\mathrm{Spec}V$ together with an isomorphism $C \cong \mathcal{C} \otimes_V Q$.

b) For a model $\mathcal{X}$ of $X$ over $V$ and a divisor $D$ in $X$ the category $\mathcal{S}_{\mathcal{X},D}$ is defined as follows: Objects are finitely presented proper $V$- morphisms $\pi : \mathcal{Y} \to \mathcal{X}$ whose generic fiber $\pi_Q : \mathcal{Y}_Q \to X$ is finite and such that $\pi_Q : \pi_Q^{-1}(X \setminus D) \to X \setminus D$ is étale. Morphisms are defined to be compatible with the structure morphism.

c) A full subcategory $\mathcal{S}^{good}_{\mathcal{X},D} \subset \mathcal{S}_{\mathcal{X},D}$ is defined by taking as objects models $\mathcal{Y}$ in $\mathcal{S}_{\mathcal{X},D}$ whose structural morphism $\lambda : \mathcal{Y} \to \mathrm{Spec}(V)$ is flat and satisfies $\lambda_* \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathrm{Spec} V}$ universally, and whose generic fiber $\lambda_Q : \mathcal{Y}_Q \to \mathrm{Spec} Q$ is smooth.

**Definition 3.2 (DeWe). -**

a) For a model $\mathcal{X}$ of $X$ over $\overline{\mathbb{Z}_p}$ and a divisor $D$ in $X$ the category $\mathfrak{B}_{\mathcal{X}_{\mathfrak{o}},D}$ is defined to be the full subcategory of $\mathbf{Vec}_{\mathcal{X}_{\mathfrak{o}}}$ consisting of vector bundles $\mathcal{E}$ on $\mathcal{X}_{\mathfrak{o}} = \mathcal{X} \otimes_{\overline{\mathbb{Z}_p}} \mathfrak{o}$ with the following property: For every $n \geq 1$ there is an object $\pi : \mathcal{Y} \to \mathcal{X}$ of $\mathcal{S}_{\mathcal{X},D}$ such that $\pi_n^* \mathcal{E}_n$ is a trivial bundle on $\mathcal{Y}_n$. Here $\pi_n, \mathcal{Y}_n$ and $\mathcal{E}_n$ are the reductions mod $p^n$ of $\pi$, $\mathcal{Y}$ and $\mathcal{E}$.

b) The full subcategory $\mathfrak{B}_{X_{\mathbb{C}_p},D}$ of $\mathbf{Vec}_{X_{\mathbb{C}_p}}$ consists of all vector bundles on $X_{\mathbb{C}_p}$ which are isomorphic to a vector bundle of the form $j^* \mathcal{E}$ with $\mathcal{E}$ in $\mathfrak{B}_{\mathcal{X}_{\mathfrak{o}},D}$ for some model $\mathcal{X}$ of $X$. Here $j$ is the open immersion of $X_{\mathbb{C}_p}$ into $\mathcal{X}_{\mathfrak{o}}$.

c) The full subcategory $\mathfrak{B}^{\#}_{X_{\mathbb{C}_p},D}$ of $\mathbf{Vec}_{X_{\mathbb{C}_p}}$ consists of all vector bundles $E$ on $X_{\mathbb{C}_p}$ such that $\alpha^*_{\mathbb{C}_p} E$ is in $\mathfrak{B}_{Y_{\mathbb{C}_p},\alpha^* D}$ for some finite covering $\alpha : Y \to X$ of $X$ by a smooth projective curve $Y$ over $\overline{\mathbb{Q}_p}$ such that $\alpha$ is étale over $X \setminus D$.

d) Finally define
$$\mathfrak{B}^s_{X_{\mathbb{C}_p}} := \bigcup_D \mathfrak{B}_{X_{\mathbb{C}_p,D}}.$$
These are are the vector bundles on $X_{\mathbb{C}_p}$ with strongly semi-stable reduction (See the Introduction of [DeWe05b] and Theorem 36)

We describe now the construction of Deninger and Werner for vector bundles in $\mathfrak{B}_{X_{\mathbb{C}_p},D}$: Now, given $\gamma$ in $Iso(F_x, F_{x'})$ (an étale path (Section 3.6) between two points $x$ and $x'$) and some $n \geq 1$, let us construct $\rho_{\mathcal{E},n}(\gamma)$. By definition of $\mathfrak{B}_{\mathcal{X}_{\mathfrak{o}},D}$ and by [DeWe05b] Corollary 3 there exists an object $\pi : \mathcal{Y} \to \mathcal{X}$ of $\mathcal{S}^{good}_{\mathcal{X},D}$ such that $\pi_n^* \mathcal{E}_n$ is a trivial bundle. Set $Y := \mathcal{Y} \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{Q}_p}$ and $V := Y \setminus \pi^* D$. Then $V \to U$ is a finite étale covering. Choose a point $y \in V(\mathbb{C}_p)$ above $x$ and let $y' = \gamma y \in V(\mathbb{C}_p)$ be the image of $y$ under the map
$$\gamma_V : F_x(V) \to F_{x'}(V).$$
Then $y'$ lies over $x'$. Since the structural morphism $\lambda : \mathcal{Y} \to \mathrm{Spec}\overline{\mathbb{Z}_p}$ satisfies $\lambda_* \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathrm{Spec}\overline{\mathbb{Z}}_p}$ universally we find $\lambda_* \mathcal{O}_{\mathcal{Y}_n} = \mathcal{O}_{\mathrm{Spec}\mathfrak{o}_n}$ and therefore the pullback map under $y_n : \mathrm{Spec}\mathfrak{o}_n \to \mathcal{Y}_n$ is an isomorphism
$$y_n^* : \Gamma(\mathcal{Y}_n, \pi_n^* \mathcal{E}_n) \tilde{\to} \Gamma(\mathrm{Spec}\mathfrak{o}_n, y_n^* \pi_n^* \mathcal{E}_n) = \Gamma(\mathrm{Spec}\mathfrak{o}_n, y_n^* \mathcal{E}_n) = \mathcal{E}_{x_n}$$

We can now define

$$\rho_{\mathcal{E},n}(\gamma) = \gamma(y)_n^* \circ (y_n^*)^{-1} = (y')_n^* \circ (y_n^*)^{-1} : \mathcal{E}_{x_n} \to \mathcal{E}_{x'_n}$$

Note that by construction $\rho_{\mathcal{E},n}$ factors over the finite set $Iso(F_x(V), F_{x'}(V))$.

**Theorem 3.3 (DeWe).** - *The preceding constructions are independent of all choices and define a continuous functor $\rho_{\mathcal{E}}$ from $\prod_1(X \smallsetminus D)$ into the category of free $\mathfrak{o}$-modules of finite rank.*

*Proof.* [DeWe05b] Theorem 22 □

This result can be extended to generic fibers:

**Theorem 3.4 (DeWe).** - *Let $X'$ be a smooth projective curve over $\overline{\mathbb{Q}_p}$ and let $f : X \to X'$ be a morphism. Let $D'$ be a divisor on $X'$.*

*a) The functor*

$$\rho : \mathfrak{B}_{X_{\mathbb{C}_p},D} \to \mathbf{Rep}_{\Pi_1(X \setminus D)}(\mathbb{C}_p)$$

*is $\mathbb{C}_p$-linear, exact and commutes with tensor products, duals, internal homs and exterior powers.*

*b) Pullback of vector bundles induces an additive and exact functor*

$$f^* : \mathfrak{B}_{X'_{\mathbb{C}_p},D'} \to \mathfrak{B}_{X_{\mathbb{C}_p},f^*D'}$$

*which commutes with tensor products, duals, internal homs and exterior powers. The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{B}_{X'_{\mathbb{C}_p},D'} & \xrightarrow{\rho} & \mathbf{Rep}_{\Pi_1(X' \setminus D')}(\mathbb{C}_p) \\ \downarrow f^* & & \downarrow A(f) \\ \mathfrak{B}_{X_{\mathbb{C}_p},f^*D'} & \xrightarrow{\rho} & \mathbf{Rep}_{\Pi_1(X \setminus f^*D')}(\mathbb{C}_p). \end{array}$$

*In particular for $E$ in $\mathfrak{B}_{X_{\mathbb{C}_p},D}$ we have*

$$\rho_{f^*E} = \rho_E \circ f_*$$

*as functors from $\Pi_1(X \setminus f^*D')$ to $\mathbf{Vec}_{\mathbb{C}_p}$.*

*c) For every automorphism $\sigma$ of $\overline{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{B}_{X_{\mathbb{C}_p},D} & \xrightarrow{\rho} & \mathbf{Rep}_{\Pi_1(X\setminus D)}(\mathbb{C}_p) \\ \downarrow{\sigma_*} & & \downarrow{\mathbf{C}_\sigma} \\ \mathfrak{B}_{{}^\sigma X_{\mathbb{C}_p},{}^\sigma D} & \xrightarrow{\rho} & \mathbf{Rep}_{\Pi_1({}^\sigma(X\setminus D))}(\mathbb{C}_p). \end{array}$$

*In particular, we have for $E$ in $\mathfrak{B}_{X_{\mathbb{C}_p},D}$ that*

$$\rho_{{}^\sigma E} = \sigma_* \circ \rho_E \circ (\sigma_*)^{-1}$$

*as functors from $\Pi_1({}^\sigma(X \setminus D))$ to $\mathbf{Vec}_{\mathbb{C}_p}$. If $X = X_K \otimes_K \overline{\mathbb{Q}_p}$ and $D = D_K \otimes_K \overline{\mathbb{Q}_p}$ for some field $\mathbb{Q}_p \subset K \subset \overline{\mathbb{Q}_p}$, so that $({}^\sigma X, {}^\sigma D)$ is canonically identified with $(X, D)$ over $\overline{\mathbb{Q}_p}$ for all $\sigma \in G_K$, the functor*

$$\rho : \mathfrak{B}_{X_{\mathbb{C}_p},D} \to \mathbf{Rep}_{\Pi_1(X\setminus D)}(\mathbb{C}_p)$$

*commutes with the left $G_K$-actions on these categories, defined by letting $\sigma$ act via $\sigma_*$ respectively via $\mathbf{C}_\sigma$.*

*Proof.* [DeWe05b] Theorem 28 □

This parallel transport can be extended to vector bundles lying in the category $\mathfrak{B}^{\#}_{X_{\mathbb{C}_p},D}$ and $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$:

**Theorem 3.5 (DeWe).** - *The preceding constructions give a well defined functor*

$$\rho : \mathfrak{B}^{\#}_{X_{\mathbb{C}_p},D} \to \mathbf{Rep}_{\Pi_1(X\setminus D)}(\mathbb{C}_p)$$

*which extends the previously defined functor. For different divisors $D$ on $X$ the corresponding functors are compatible and define a functor*

$$\rho : \mathfrak{B}^s_{X_{\mathbb{C}_p}} = \bigcup_D \mathfrak{B}_{X_{\mathbb{C}_p},D} \to \mathbf{Rep}_{\Pi_1(X)}(\mathbb{C}_p)$$

*Proof.* [DeWe05b] Proposition 32 and Theorem 36. As it will be used later we sketch the basic part of the construction: Choose a ramified Galois covering $\alpha : Y \to X$ which is étale over $X \setminus D$ such that $\alpha^* E$ lies in $\mathfrak{B}_{Y_{\mathbb{C}_p},\alpha^* D}$. For an étale path $\gamma$ from $x_1$ to $x_2$ in $X \setminus D$ we set

$$\rho_E(\gamma) = \rho_{\alpha^* E}(\gamma') : E_{x_1} = (\alpha^* E)_{y_1} \to (\alpha^* E)_{y_2} = E_{x_2}.$$

where $y_1 \in V(\mathbb{C}_p)$ $(V := Y \setminus \alpha^* D)$ lies above $x_1$ and $\gamma'$ is the unique path in $V$ with $\alpha_* \gamma' = \gamma$ from $y_1$ to a point $y_2$ above $x_2$. □

## 3.3 Deninger-Werner parallel transport on abelian varieties

| | |
|---|---|
| $A$ | an abelian variety over $\overline{\mathbb{Q}_p}$ having good reduction |
| $\mathcal{A}/\overline{\mathbb{Z}_p}$ | an abelian scheme over $\overline{\mathbb{Z}_p}$ with generic fiber $A$ |
| $x = \mathrm{Spec}\overline{\mathbb{Q}_p} \to A$ | the zero section of $A$ |

We will review the étale parallel transport of Deninger and Werner in the case of abelian varieties having good reduction [DeWe05a].

**Definition 3.6 (DeWe).** -

a) Let $\mathfrak{B}_{\mathcal{A}_\mathfrak{o}}$ be the full category of the category of vector bundles on the abelian scheme $\mathcal{A}_\mathfrak{o}$ consisting of all vector bundles $E$ on $\mathcal{A}_\mathfrak{o}$ satisfying the following property: For all $n \geq 1$ there exists some $N = N(n) \geq 1$ such that the reduction $(N^*E)_n$ of $N^*E$ modulo $p^n$ is trivial on $\mathcal{A}_n = \mathcal{A} \otimes_{\overline{\mathbb{Z}_p}} \mathfrak{o}_n$. Here $N : \mathcal{A} \to \mathcal{A}$ denotes the multiplication by $N$.

b) Let $\mathfrak{B}_{A_{\mathbb{C}_p}}$ be the full subcategory of the category of vector bundles on $A_{\mathbb{C}_p}$ consisting of all vector bundles $F$ on $A_{\mathbb{C}_p}$ which are isomorphic to the generic fiber of a vector bundle $E$ in the category $\mathfrak{B}_{\mathcal{A}_\mathfrak{o}}$.

We will now sketch how to attach a $p$-adic representation of the Tate-module $TA$ of $A$ to a vector bundle $F$ in $\mathfrak{B}_{A_{\mathbb{C}_p}}$:

Let $E$ be a vector bundle in $\mathfrak{B}_{\mathcal{A}_\mathfrak{o}}$ with generic fiber $F$. Fix some $n \geq 1$. Then there exists some $N = N(n) \geq 0$, such that the reduction $(N^*E)_n$ is trivial on $\mathcal{A}_n$. The structure morphism $\lambda : \mathcal{A}_\mathfrak{o} \to \mathrm{Spec}\mathfrak{o}$ satisfies $\lambda_* \mathcal{O}_{\mathcal{A}_\mathfrak{o}} = \mathcal{O}_{\mathrm{Spec}\mathfrak{o}}$ universally. Hence $\Gamma(\mathcal{A}_n, \mathcal{O}) = \mathfrak{o}_n$, and therefore the pullback map

$$x_n^* : \Gamma(\mathcal{A}_n, (N^*E)_n) \mapsto \Gamma(\mathrm{Spec}\mathfrak{o}_n, x_n^* E_n) = E_n$$

is an isomorphism of free $\mathfrak{o}_n$-modules. (Note that $N \circ x_n = x_n$) On $\Gamma(\mathcal{A}_n, (N^*E)_n)$ the group $A_N(\overline{\mathbb{Q}_p})$ acts in a natural way by translation. Define a representation $\rho_{E,n} : TA \to Aut_{\mathfrak{o}_n}(E_{x_n})$ as the composition:

$$\rho_{E,n} : TA \twoheadrightarrow A_N(\overline{\mathbb{Q}_p}) \longrightarrow Aut_{\mathfrak{o}_n}(\Gamma(\mathcal{A}_n, (N^*E)_n)) \xrightarrow[x_n^*]{\sim} Aut_{\mathfrak{o}_n} E_{x_n}$$

The representations $\rho_{E,n}$ form a projective system and give a well defined representation $\rho_E : TA \to Aut(E_x)$. Taking generic fibers one obtains:

**Theorem 3.7 (DeWe).** -

*a) The category $\mathfrak{B}_{A_{\mathbb{C}_p}}$ is closed under direct sums, tensor products, duals, internal homs and exterior powers. It contains all line bundles of degree 0. Besides, it is closed under extensions, i.e. if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of vector bundles on $A_{\mathbb{C}_p}$ such that $F'$ and $F''$ are in $\mathfrak{B}_{A_{\mathbb{C}_p}}$, then $F$ is also contained in $\mathfrak{B}_{A_{\mathbb{C}_p}}$.*

*b) The association $F \mapsto \rho_F$ defines an additive exact functor*

$$\rho : \mathfrak{B}_{A_{\mathbb{C}_p}} \to \mathbf{Rep}_{TA}(\mathbb{C}_p),$$

*where $\mathbf{Rep}_{TA}(\mathbb{C}_p)$ is the category of continuous representations of $TA$ on finite dimensional $\mathbb{C}_p$-vector spaces. This functor commutes with tensor products, duals, internal homs and exterior powers.*

*c) Let $f : A \to A'$ be a homomorphism of abelian varieties over $\overline{\mathbb{Q}_p}$ with good reduction. Then pullback of vector bundles induces an additive exact functor*

$$f^* : \mathfrak{B}_{A'_{\mathbb{C}_p}} \to \mathfrak{B}_{A_{\mathbb{C}_p}},$$

*which commutes with tensor products, duals internal homs and exterior powers (up to canonical identifications). The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{B}_{A'_{\mathbb{C}_p}} & \xrightarrow{f^*} & \mathfrak{B}_{A_{\mathbb{C}_p}} \\ \rho \downarrow & & \downarrow \rho \\ \mathbf{Rep}_{TA'}(\mathbb{C}_p) & \xrightarrow{F} & \mathbf{Rep}_{TA}(\mathbb{C}_p) \end{array}$$

*where $F$ is the functor induced by composition with $Tf : TA \to TA'$.*

*d) Assume that $A$ is defined over a finite extension $K$ of $\mathbb{Q}_p$ and set $G_K = Gal(\overline{\mathbb{Q}_p}/K)$. For every $\sigma \in G_K$ the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{B}_{A_{\mathbb{C}_p}} & \xrightarrow{\rho} & \mathbf{Rep}_{TA}(\mathbb{C}_p) \\ g_\sigma \downarrow & & \downarrow \sigma_* \\ \mathfrak{B}_{A_{\mathbb{C}_p}} & \xrightarrow{\rho} & \mathbf{Rep}_{TA}(\mathbb{C}_p) \end{array}$$

*where the functor $g_\sigma$ maps $F$ to ${}^\sigma F$.*

*e) If $A$ is defined over a finite extension $K$ of $\mathbb{Q}_p$, and if $\hat{A}$ denotes the dual abelian variety, that classifies line bundle of degree 0 on $A$, then the association $F \mapsto \rho_F$ induces an isomorphism of topological groups*

$$\hat{A}(\mathbb{C}_p) \cong CH^\infty(TA).$$

*Here $CH^\infty(TA)$ denotes the group of continuous characters $\chi : TA \to \mathbb{C}_p^*$ whose stabilizer in $G_K = Gal(\overline{\mathbb{Q}_p}/K)$ is open.*

*Proof.* [DeWe05a] Theorem 1 and Proposition 1 □

There is also a relation with the functor on curves: Let $X$ be a smooth irreducible projective curve over $\overline{\mathbb{Q}_p}$ which has good reduction. Fix a point $x \in X(\overline{\mathbb{Q}_p})$ and denote by $\pi_1(X, x)$ the algebraic fundamental group with base point $x$. Let $f : X \to A := Jac(X)$ be the embedding induced by $x \mapsto 0$.

**Proposition 3.8.** *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{B}_{A_{\mathbb{C}_p}} & \xrightarrow{f^*} & \mathfrak{B}_{X_{\mathbb{C}_p}} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ \mathbf{Rep}_{TA}(\mathbb{C}_p) & \xrightarrow{\tilde{f}} & \mathbf{Rep}_{\pi_1(X,x)}(\mathbb{C}_p) \end{array}$$

*where $\tilde{f}$ is the functor induced by composition with the homomorphism $f_* : \pi_1(X, x) \to TA$.*

*Proof.* [DeWe05a] Lemma 2 □

## 3.4 Faltings' $p$-adic Simpson correspondence

| | |
|---|---|
| $K$ | a finite extension of $\mathbb{Q}_p$ |
| $V$ | the ring of integers in $K$ |
| $\mathcal{X}$ | a proper $V$-scheme |

In [Fal05] Faltings constructs a $p$-adic analogue of the correspondence described by Simpson and Corlette [Sim92]. We sketch his results briefly:

**Definition 3.9 (Hitchin, Simpson).** A *Higgs bundle* on an algebraic variety $Y$ over a field $L$ is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a vector bundle on $Y$ and $\theta$ an element of $End(\mathcal{E}) \otimes \Omega^1_{Y/L}$ satisfying $\theta \wedge \theta = 0$ (In the case that $Y$ is a curve this condition is superfluous). The morphism $\theta$ is called a *Higgs field.*

**Remark 3.10.** -

a) The scheme $\mathcal{X}$ is supposed to have *"toroidal singularities"*, because Faltings' *almost purity theorem* ([Fal02] Theorem 4) is stated for schemes having this kind of singularities. Examples of such schemes are schemes that are *smooth* or *semi-stable*.

b) Faltings defines a sheaf $\overline{\mathcal{O}}$ on a suitable situs on $\mathcal{X} \setminus D$ for some divisor $D$. He calls a vector bundle over $\overline{\mathcal{O}}/p^s$ (for some $s \in \mathbb{Q}$, $s > 0$) a *"generalized representation"*. Locally (affine) such generalized representations are given by projective modules over a ring $\overline{R}$ equipped with a semi-linear action of the étale fundamental group of the generic fiber.

c) We restrict us for simplicity to a field $K$ that is a finite extension of $\mathbb{Q}_p$, whereas Faltings considers a more general situation.

Faltings established the following $p$-adic analogue of Simpsons' correspondence for Higgs-bundles on $\mathcal{X}_{\mathbb{C}_p}$ in the case of curves, i.e. $\mathcal{X}_{\mathbb{C}_p}$ is a curve:

**Theorem 3.11 (Faltings).** *There exists an equivalence of categories between Higgs-bundles and generalized representations, if we allow $\mathbb{C}_p$ coefficients.*

*Proof.* [Fal05] Theorem 6, Section 2 □

**Remark 3.12.** -

a) The equivalence depends on the choice of a $p$-adic exponential function for the multiplicative group, and a lift of $\mathcal{X}$ to the dual numbers $A_2$ of a Fontaine ring ([Fal05] Section 1).

b) The category of generalized representations contains the category of representations of the étale fundamental group of $\mathcal{X}_{\overline{\mathbb{Q}_p}}$ on free finitely generated $\mathfrak{o}$-modules $\mathbb{L}$ (étale local systems) as a full subcategory ([Fal05] Section 2).

a) The constructions of Faltings and Deninger-Werner are assumed to coincide for Higgs bundles $(\mathcal{E}, \theta)$ with $\mathcal{E} \in \mathfrak{B}^s_{\mathcal{X}_{\mathbb{C}_p}}$ and $\theta = 0$ ([Fal05] Section 5).

b) It is difficult to characterize the image of Faltings correspondence, i.e. which Higgs bundles correspond to actual representation. It is known that the (Higgs) vector bundles in the image are all semi-stable of slope 0 and all rank one Higgs bundles are in the image ([Fal05] Section 5).

c) It is an interesting question which representations of the étale fundamental group correspond to Higgs-bundles with zero Higgs field i.e. $\theta = 0$.

We now state a key lemma used by Faltings to attach a Higgs bundle to a generalized representation. The construction is affine, and Faltings uses in [Fal05] Section 3 the following notation (in the case that $\mathcal{X}$ is a curve): Let Spec$R \subset \mathcal{X}$ be a *small* affine, that is $R$ is étale over over a toroidal model (e.g. $R$ is étale over $V[x]$ (smooth) or étale over $V[x,y]/(xy-p)$ (semistable)). Denote by $\overline{R}$ the integral closure in the maximal étale extension of $U_K^\circ = U_K \setminus D_K$ ($D$ a divisor as above) where $U_K := \mathrm{Spec}(R \otimes_V K)$. Denote by $R_\infty \subset \overline{R}$ the sub-extension obtained by adjoining roots of characters of the torus (e.g roots of $x$ (smooth case)) and set $R_1 := R \otimes_V \overline{V}$. Set $\Delta := Gal(\overline{R}/R_1)$ and $\Delta_\infty = Gal(R_\infty/R_1)$. For a more detailed discussion we refer the reader to [Fal02] Section 2c page 205, [Ols06] 3.5 and [Len97] Section 6 Corollary 17.

**Lemma 3.13 (Faltings).** - *Suppose that $\alpha > 1/(p-1)$ is a rational number, and $\overline{M} \cong \overline{R}^r/(p^s)$ is a generalized representation (it admits a semi-linear $\Delta$ operation).*

*a) Suppose that $\overline{M}$ is trivial modulo $p^{2\alpha}$. Then its reduction modulo $p^{s-\alpha}$ is given by a representation $\Delta_\infty \to GL(r, R_1/(p^{s-\alpha}))$, and this representation is trivial modulo $p^\alpha$.*

*b) Suppose given two representations $\Delta_\infty \to GL(r_i, R_1/(p^{s-\alpha}))$ (i=1,2), trivial modulo $p^\alpha$, and a $\overline{R}-\Delta$-linear map between the associated generalized representations. Then its reduction modulo $p^{s-\alpha}$ is given by an $R_1 - \Delta_\infty$-linear map of representations.*

*Proof.* [Fal05] Lemma 1. Faltings uses standard group cohomology to find such representations. To compute certain cohomology groups appearing in the construction, one needs to use Fatlings method of almost étale extensions □

**Remark 3.14.** -

a) Let $\Delta_\infty \to Aut_{R_1/(p^{s-\alpha})}(M)$ be a representations as in 3.13 a), on a free $R_1/(p^{s-\alpha})$-module $M$. Then in [Fal05] Remark ii) associates a Higgs-field $\theta$ to this representation by applying the logarithm map to the images of generators of $\Delta_\infty$ (This is possible because the logarithm converges for arguments divisible by $p^\alpha$). The resulting Higgs-field $\theta$ is an element of $End(M) \otimes \tilde{\Omega}^1_{R/V} \otimes \overline{V}(-1)$ with commuting components.

b) If this representation is trivial then $log(1) = 0$ and the resulting Higgs-field $\theta$ equals zero.

c) Faltings can extend this result to $p$-adic representations by using the inductive method of liftings.

## 3.5 Unit-root $F$-crystals

| | |
|---|---|
| $K$ | a finite extension of $\mathbb{Q}_p$ |
| $V$ | its ring of integers |
| $k$ | the residue field of $V$ |
| $W$ | the ring of Witt vectors of $k$ |
| $\pi$ | a uniformizer in $V$ |
| $\sigma$ | a lifting (to $V$) of the $q$-power Frobenius on $k$ leaving $\pi$ invariant |
| $\mathcal{X}_k$ | a smooth $k$-scheme |
| $\mathcal{X}/W$ | a formally smooth lifting of $\mathcal{X}_k$ |
| $\phi$ | a lifting (to $\mathcal{X}$) of the absolute Frobenius on $\mathcal{X}_k$ |
| $\overline{x}$ | a geometric point of $\mathcal{X}_k$ |

In [Katz73] Katz describes a correspondence between representations of the étale fundamental group of a smooth scheme in characteristic $p$ and formal vector bundles equipped with a Frobenius morphism. Crew describes in [Crew87] the image of this correspondence in certain cases. We will reproduce some of their results here:

**Definition 3.15 (Crew).** A $F$-lattice on $\mathcal{X}/(V, \phi)$ is a locally free $V \otimes \mathcal{O}_{\mathcal{X}}$-module $M$ endowed with a map

$$\Phi : \phi^* M \to M$$

such that $\Phi \otimes \mathbb{Q}$ is an isomorphism. If $\Phi$ is an isomorphism, then $(M, \Phi)$ is called an unit-root $F$-lattice.

**Theorem 3.16 (Crew, Katz).** *There is a natural equivalence of categories*

$$H : Rep_{V^\sigma} \pi_1(X, \overline{x}) \cong \textit{ (unit-root } F\textit{-lattices on } \mathcal{X}/V\textit{)}$$

*Proof.* ([Katz73] Proposition 4.1.1 page 74, [Crew87] Theorem 2.2). We reproduce parts of [Crew87] Theorem 2.2: Let $\rho : \pi_1(\mathcal{X}_k, \overline{x}) \to Aut_{V^\sigma}(\mathbb{L})$ be a continuous representation on a finite free $V^\sigma$-module $\mathbb{L}$. For $n \geq 1$ let $\mathcal{X}_n = \mathcal{X} \otimes W_n$ and let $G_n$ be the image of $\pi_1(X, \overline{x})$ in $Aut_{V^\sigma}(\mathbb{L}/p^n\mathbb{L})$. The homomorphism $\pi_1(X, \overline{x}) \to G_n$ classifies an étale cover $\mathcal{Y}_{k,n} \to \mathcal{X}_k$ which has a unique étale lifting $\mathcal{Y}_n \to \mathcal{X}_n$; the action of $G_n$ on $\mathcal{Y}_{k,n}$ extends uniquely to

$\mathcal{Y}_n$, as does the action of $\phi$ on $\mathcal{X}_n$. The "opposite" action makes $\mathcal{O}_{\mathcal{Y}_{k,n}}$ into a right $G_n$-module and we let

$$M_n := \pi_{n*}\mathcal{O}_{\mathcal{Y}_n} \otimes_{W_n[G_n]} \mathbb{L}_n, \qquad M := \varprojlim M_n.$$

By uniqueness, the action of $G_n$ on $\mathcal{O}_{\mathcal{Y}_n}$ commutes with $\phi$, so that the mapping $\Phi = \phi \otimes id$ gives compatible isomorphisms $\Phi : \phi^* M_n \cong M_n$, whence $\Phi : \phi^* M \cong M$. This $(M, \Phi)$ is $H(\rho)$. The inverse functor for $H$ is given by mapping $(M, \Phi)$ to "$Ker(1 - \Phi)$". More precisely, for $n \geq 1$ the group $G_n$ acts on the finite dimensional $V^\sigma/p^n$ module $\mathbb{L}_n = Ker(1_n - \Phi_n)$, where $1_n - \Phi_n : \pi_n^* M_n \to \pi_n^* M_n$. Then $\pi_1(X, \overline{x})$ acts on the inverse limit

$$\mathbb{L} = \varprojlim \mathbb{L}_n$$

□

## 3.6 Various fundamental groups

| | |
|---|---|
| $K$ | a complete ultrametric field $(K, \vert\ \vert)$ |
| $V$ | the ring of integers in $K$ |
| $S$ | a $K$-analytic space (Berkovich analytic space) |

In this section we recall some facts about fundamental groups of Berkovich analytic spaces and schemes. Our sources are [Gro71], Andres' book [And03] Chapter III, [deJ95], [Her05] and [Len97].

**Definition 3.17 (André).** - A paracompact strictly $K$-analytic space is called a *$K$-manifold*, if for any $s \in S$ there is a neighborhood $U(s)$ of $s$ which is isomorphic to an affinoid subdomain of some smooth space.

From now on we will assume that $S$ is a $K$-manifold.

**Remark 3.18 (Berkovich, André, deJong).** -

a) If $X$ is a $K$ variety, then $X$ is smooth if and only if $X^{an}$ is a $K$-manifold.

b) $K$-manifolds are locally arcwise-connected and locally simply connected, hence any pointed $K$-manifold $(S, s)$ admits a universal covering $(\tilde{S}, \tilde{s})$.

c) Let $X$ be a smooth projective algebraic curve over $K$. Assume that $X^{an}$ has a semi-stable formal model $\mathfrak{X}$ (what is true if e.g. $K$ is algebraically closed). Then $\pi_1^{top}(X^{an}, \overline{x})$ is isomorphic to the fundamental group of the graph associated to the special fiber of $\mathfrak{X}$. This group does not depend on the choice of the semi-stable model ([deJ95] Proposition 5.3)

d) A geometric point $\overline{s}$ of $S$ is a point with value in some complete algebraically closed extension $(\Omega, |\ |)$ of $(K, |\ |)$.

**Definition 3.19 (Berkovich, De Jong, Herz).** - A morphism $f : S' \to S$ is a *covering* (resp. *étale* covering, resp. *topological* covering, resp. *finite topological* covering) if $S$ is covered by open subsets $U$ such that $f^{-1}U$ is a disjoint union of open subsets $V_j \subset S'$ (i.e. $f^{-1}U = \coprod_{j \in J} V_j$), such that $f$ restricted to each $V_j$ is finite (resp. étale finite , resp. an isomorphism, resp. an isomorphism and $f : S' \to S$ is finite). The categories of étale coverings, finite étale covering, topological coverings, topological finite coverings will be denoted by $Cov_S^{et}$, $Cov_S^{alg}$, $Cov_S^{top}$, $Cov_S^{ftop}$.

**Definition 3.20 (André).** - An étale covering $S' \to S$ is called *temperate* if it is a quotient ([And03] III Lemma 1.2.8) of a composite étale covering $T' \to T \to S$, where $T' \to T$ is a topological covering, and $T \to S$ is a finite étale covering.

I.e. there is a commutative diagram

$$\begin{array}{ccc} T' & \longrightarrow & S' \\ {\scriptstyle top}\downarrow & & \downarrow{\scriptstyle et} \\ T & \xrightarrow{alg} & S \end{array}$$

with $T' \in Cov_T^{top}$, $T \in Cov_S^{alg}$, $S' \in Cov_S^{et}$.

**Remark 3.21 (André, Herz, deJong).** - There are inclusions (fully faithful embeddings)

$$Cov_S^{alg} \hookrightarrow Cov_S^{et}, \qquad Cov_S^{top} \hookrightarrow Cov_S^{et}, \qquad Cov_S^{ftop} \hookrightarrow Cov_S^{top},$$

$$Cov_S^{top} \hookrightarrow Cov_S^{temp}, \qquad Cov_S^{alg} \hookrightarrow Cov_S^{temp}, \qquad Cov_S^{temp} \hookrightarrow Cov_S^{et}.$$

Let $\overline{s}$ be a geometric point of $S$. Consider the fiber functor

$$F_{S,\overline{s}}^{et} = F_{\overline{s}}^{et} : Cov_S^{et} \to Sets$$

$$S' \mapsto \text{geometric points } \overline{s}' \text{ of } S' \text{ above } \overline{s}$$

and its restrictions

$$F_{\overline{s}}^{alg} : Cov_S^{alg} \to Sets;$$
$$F_{\overline{s}}^{top} : Cov_S^{top} \to Sets;$$
$$F_{\overline{s}}^{ftop} : Cov_S^{ftop} \to Sets;$$

$$F_{\overline{s}}^{temp} : Cov_S^{temp} \to Sets.$$

An *étale path* from $\overline{s}$ to another geometric point $\overline{t}$ of $S$ is an isomorphism of fiber functors $F_{\overline{s}}^{et} \cong F_{\overline{t}}^{et}$. The set of étale path is topologized by taking as fundamental open neighborhoods of an étale path $\alpha$ the set $Stab_{S',\overline{s}'} \circ \alpha$, where $Stab_{S',\overline{s}'}$ runs among the stabilizers in $Aut(F_{\overline{s}}^{et})$ of arbitrary geometric points $\overline{s}'$ above $\overline{s}$ in arbitrary étale coverings $S'/S$.

One can define various fundamental groups of $K$-manifolds:

Let $Cov_S^{\bullet}$ be a full subcategory of $Cov_S^{et}$ which is stable under taking connected components, fiber products and quotients. Examples are $Cov_S^{et}$, $Cov_S^{alg}$, $Cov_S^{top}$, $Cov_S^{ftop}$, $Cov_S^{temp}$. Denote by $F_{\overline{s}}^{\bullet}$ the restriction of $F_{\overline{s}}^{et}$ to $Cov_S^{\bullet}$, and set

$$\pi_1^{\bullet}(S, \overline{s}) = Aut F_{\overline{s}}^{\bullet}$$

where the topology is as above with $\overline{t} = \overline{s}$.

**Lemma 3.22.** *The natural map*

$$\pi_1^{\bullet}(S, \overline{s}) \to \lim_{\substack{\leftarrow \\ Stab^{\bullet}_{S',\overline{s}'}}} \pi_1^{\bullet}(S, \overline{s})/Stab^{\bullet}_{S',\overline{s}'}$$

*is a homeomorphism. In particular $\pi_1^{\bullet}(S, \overline{s})$ is a separated pro-discrete topological space.*

*Proof.* [And03] III Lemma 1.4.2 □

**Proposition 3.23.** *Let $Cov_S^{\bullet\bullet}$ be a full subcategory of $Cov_S^{\bullet}$ stable under taking connected components, fiber products and quotients. Then the continuous homomorphism*

$$\pi_1^{\bullet}(S, \overline{s}) \to \pi_1^{\bullet\bullet}(S, \overline{s})$$

*has dense image.*

*Proof.* [And03] III Corollary 1.4.8. □

The following remark relates the fundamental group of a variety with automorphisms of finite étale Galois coverings.

**Remark 3.24 ([Gro71]).** - Let $Z$ be a variety over $\overline{\mathbb{Q}_p}$ and choose a geometric point $z$ in $Z(\mathbb{C}_p)$. Let $F_z$ be the fiber functor from the category of finite étale coverings $Z'$ of $Z$ to the category of finite sets defined by $F_z := Mor_Z(z, \_)$. The functor $F_z$ is strictly pro-representable: There is a projective system $\tilde{Z} = (Z_i, z_i, \varphi_{ij})_{i \in I}$ of pointed Galois coverings of $Z$ where $I$ is a directed set, and the $z_i \in Z_i(\mathbb{C}_p)$ are points over $z$. Moreover, for $i \geq j$

the map $\varphi_{ij} : Z_i \to Z_j$ is an epimorphism over $Z$ such that $\varphi_{ij}(z_i) = z_j$ and such that the natural map

$$\lim_{\overrightarrow{i}} Mor_Z(Z_i, Z') \to F_z(Z')$$

induced by evaluation on the $z_i$'s is a bijection for every $Z'$.

There is an isomorphism of topological groups

$$\pi_1(Z, z) = Aut(F_z) \to (\lim_{\overrightarrow{i}} Aut_Z(Z_i))^{op}.$$

Here the natural transformation $\sigma_{F_z} : F_z \to F_z$ given by the family of compatible bijections $\sigma_{F_z(Z_i)} : F_z(Z_i) \cong F_z(Z_i)$ for $(i \in I)$ is sent to the projective system $(\sigma_i)_{i \in I}$ where $\sigma_i \in Aut_Z(Z_i)$ is uniqueley defined by the relation:

$$\sigma_i(z_i) = \sigma_{F_z(Z_i)}(z_i).$$

Let $Y/Z$ be a Galois étale cover with group $G := Aut_Z Y$. Choose a point $y$ in $Y(\mathbb{C}_p)$ lying over $y$. It determines a map of projective systems $\tilde{Z} \to Y$, represented by a morphism $a_i : Z_i \to Y$ over $Z$ wit $z_i \mapsto y$. Consider the induced epimorphism

$$\psi_i : Aut_Z Z_i \twoheadrightarrow Aut_Z Y$$

defined by $\psi_i(\sigma)(y) = a_i \circ \sigma \circ z_i$. The composition

$$\varphi_y : \pi_1(Z, z) \twoheadrightarrow Aut_Z^{op} Z_i \twoheadrightarrow G^{op}$$

depends only on $y$, but not on $i$. For two different choices $y, y'$ of points in $Y(\mathbb{C}_p)$ over $z$, we obtain the corresponding epimorphisms $a_i$, $a_i'$ and $\psi_i, \psi_i'$. Because $Y/Z$ is Galois, there is a $\tau \in G$ satisfying $\tau y = y'$. One can check, that the two epimorphisms $\psi_i$ and $\psi_i'$ satisfy the relation

$$\psi_i'(\sigma) = \tau \psi_i(\sigma) \tau^{-1} \qquad \text{for all } \sigma \in Aut_Z Z_i$$

in other words, they are conjugated.

**Remark 3.25.** In [Gro71] Grothendieck developed the theory of the étale fundamental group for schemes that are locally noetherian. We will work with schemes that are defined over $\mathfrak{o}$ or $\overline{\mathbb{Z}_p}$, both rings are not noetherian. Grothendiecks' theory was extended to arbitrary (not necessarily locally noetherian) connected schemes in [Len97]. We assume that an important reason to restrict to the locally noetherian case in [Gro71] was that Grothendiecks' existence theorem was only stated for noetherian formal schemes, which played an crucial role in [Gro71] Expose X. It would be interesting to generalize Expose X to the non noetherian situation (See Section 3.7 for some available theorems).

## 3.7 GAGA for vector bundles on formal schemes

| | |
|---|---|
| $V$ | a valuation ring of Krull dimension 1 complete and separated |
| $(\pi)$ | an ideal generated by an element $\pi \neq 0$ of the maximal ideal |

In this section we want to collect some information about vector bundles defined over $\mathfrak{o}$ on a projective scheme $\mathcal{X}$ defined over $\mathcal{O}_K$ (the ring of integers of a finite extension $K$ of $\mathbb{Q}_p$) and their formal completions. We want to know if a projective systems of vector bundles modulo $p^n$ defines an algebraic vector bundle. In the noetherian case this is already in Grothendieck [EGA] I 10 (Schémas formels) and III . Most properties carry over to schemes that are topologically of finite presentation over the ring $\mathfrak{o}$ which is not noetherian.

**Definition 3.26 ([BoLu93]).** - An $V$-algebra $A$ is called *topologically of finite type* (*tf type*) if it is isomorphic to a quotient $V\langle\xi\rangle/\mathfrak{a}$, where $\xi$ is a finite set of variables and were $\mathfrak{a} \subset V\langle\xi\rangle$ is an ideal. If in addition $\mathfrak{a}$ is finitely generated, we call $A$ of *topologically finite presentation* (tf presentation). An $V$-algebra of tf presentation is called admissible, if it has no $(\pi)$-torsion.

**Remark 3.27.** The standard example for the valuation ring $(V, \pi)$ is $(\mathfrak{o}, p)$. If $A'$ is a $\mathcal{O}_K$-algebra of finite type, then $A := \widehat{A' \otimes_{\mathcal{O}_K}} \mathfrak{o}$ is an $\mathfrak{o}$ algebra of tf presentation.

**Proposition 3.28.** *Let $A$ be an $V$-algebra of tf presentation. Then $A$ is a coherent ring; in particular each $A$-module of finite presentation is coherent.*

*Proof.* [BoLu93] Proposition 1.3 □

**Proposition 3.29 (Theorem A for formal schemes).** *Let $\mathcal{M}$ be a $\mathcal{O}_{\mathfrak{X}}$-module on $\mathfrak{X} = SpfA$, where $A$ is an $V$-algebra of tf presentation. Then $\mathcal{M}$ is coherent if and only if there is a coherent $A$-module $M$ such that as an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is isomorphic to the $\mathcal{O}_{\mathfrak{X}}$-module $M^{\triangle}$ associated to $M$.*

*This $A$-module $M$ is uniquely determined by $\mathcal{M}$ up to $A$-module isomorphism.*

*Proof.* [Ull95] Proposition 2.3 □

**Proposition 3.30 (Theorem B for formal schemes).** *Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module on $\mathfrak{X} = SpfA$, where $A$ is an $V$-algebra of tf presentation. Then*

$$H^q(\mathfrak{X}, \mathcal{M}) = 0 \qquad \textit{for all } q > 0.$$

*Proof.* [Ull95] Proposition 5.1 □

**Proposition 3.31.** *Let $A$ be a finitely generated or a topologically finitely generated $V$-algebra. Then each finitely generated $A$-module without $(\pi)$-torsion is coherent over $A$.*

*Proof.* [Ull95] Proposition 1.6 □

We will need the following three GAGA theorems for formal schemes from [Ull95] Theorem 6.5:

**Theorem 3.32 (1st GAGA).** - *Let $X$ be a proper Spec$V$ scheme. Assume that $\hat{X}$, the formal scheme associated to $X$ is (locally) of topologically finite presentation over $V$. Let $\mathcal{M}$ be a $\mathcal{O}_X$-module of finite presentation. Then for each $q \in \mathbb{Z}$ one has canonical isomorphisms of $V$-modules*

$$H^q(X, \mathcal{M}) \cong H^q(\hat{X}, \hat{\mathcal{M}}).$$

*Proof.* [Ull95] Theorem 6.4. The theorem there is originally stated with some restrictions on the sheafs. These restrictions are not necessary by a theorem of Gabber: [Fuj95] Proposition 1.2.3, see also [Ull95] "Note added in proof" in the end of this paper □

**Theorem 3.33 (2nd GAGA).** - *Let $X$ be a proper Spec$V$ scheme. Assume that $\hat{X}$ is (locally) of topologically finite presentation over $V$. Let $\mathcal{M}$, $\mathcal{F}$ be finitely presented $\mathcal{O}_X$-module. Then one has a canonical isomorphism of $V$-modules*

$$Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{M}) \cong Hom_{\mathcal{O}_{\hat{X}}}(\hat{\mathcal{F}}, \hat{\mathcal{M}})$$

*Proof.* [Ull95] Theorem 6.5. The theorem there is originally stated with some restrictions on the sheafs. These restrictions are not necessary by a theorem of Gabber: [Fuj95] Proposition 1.2.3, see also [Ull95] "Note added in proof" in the end of this paper □

**Theorem 3.34 (3rd GAGA).** - *Let $X$ be a projective Spec$V$ scheme. Assume that $\hat{X}$ is (locally) of topologically finite presentation over $V$. Then for each coherent $\mathcal{O}_{\hat{X}}$-module $\mathcal{M}$ there is a finitely presented $\mathcal{O}_X$-module $\mathcal{M}'$ whose completion $\widehat{\mathcal{M}'}$ is isomorphic to $\mathcal{M}$.*

*Proof.* [Ull95] Theorem 6.8 □

**Lemma 3.35 (Gabber).** - *Let $A'$ be a finitely generated algebra over $V$. For a finitely generated $A'$-algebra $B$ the $\pi$-adic completion $\hat{B}$ is flat over $B$.*

*Proof.* [Fuj95] Proposition 1.2.3 □

**Corollary 3.36.** *Assumptions are as in Lemma 3.35. For a finitely generated module $M$ with the $\pi$-adic topology, the Artin-Rees lemma is valid and $\hat{M} = M \otimes_B \hat{B}$.*

*Proof.* [Fuj95] Corollary 1.2.7 □

**Corollary 3.37.** *Assumptions are as in Lemma 3.35. Let $M$, $N$ be two $B$-modules of finite type. Then there are canonical isomorphisms*

$$\widehat{M \otimes_B N} \cong \hat{M} \otimes_{\hat{B}} \hat{N}$$

*If furthermore $M$ is finitely presented then*

$$\widehat{Hom_B(M, N)} \cong Hom_{\hat{B}}(\hat{M}, \hat{N})$$

*Proof.* For noetherian rings this is shown in [EGA] Chap. $0_{new}$ Corollaire 7.3.7. The proof works as well using Lemma 3.35 instead. □

**Corollary 3.38.** *Assumptions are as in Lemma 3.35. Let $M$ be a $B$-module of finite type. Assume that $\hat{M}$ is a projective $\hat{B}$-module. Then $M$ is a projective $B$-module.*

*Proof.* Use Corollary 3.37 and the universal property of projective modules □

**Lemma 3.39.** *Let $A$ be a ring, $\mathcal{I}$ an ideal of $A$, such that $A$ is separated and complete for the $\mathcal{I}$-preadic topology. Set $A_n = A/\mathcal{I}^{n+1}$, and let $M_n$ be a projective system of $A_n$-modules, such that for all n the homomorphism $M_{n+1} \otimes_{A_{n+1}} A_n \to M_n$ induced by the di-homomorphism of transition $M_{n+1} \to M_n$ is bijective. Suppose that the $M_n$ are projective and $M_0$ is of finite type. Then $M = \lim\limits_{\leftarrow} M_n$ is an projective $A$-module of finite type, such that the canopnical morphism $M \otimes_A A_0 \to M_0$ is bijective.*

*Proof.* [EGA] IV Quatrieme Partie, Lemme (18.3.2.1) □

## 3.8 The universal topological covering of a curve

| | |
|---|---|
| $K$ | a finite extension of $\mathbb{Q}_p$ |
| $V$ | its ring of integers |
| $k$ | the residue field |

In this section we recall the definition and the construction of the *universal topological (analytic) covering* of a rigid curve. For our comparison between Berkovich and DeWe parallel transport we need to work in the language of formal schemes. Thus we need also formal models of our curve and its universal covering. For the relation between formal schemes, rigid spaces and analytic spaces we refer the reader to [BoLu93], [FrPu04] Sections 4,5, [Ber93] 1 and [Ber94] Section 1. We follow the exposition in [FrPu04] Section 5.7 and [Co00] Section 1.

**Definition 3.40 (van der Put).** - Let $X$ be a geometrically connected rigid space over a complete non-archimedean field $L$.

a) A *trivial covering* of $X$ is a morphism of rigid spaces $\phi : Y \to X$ such that the restriction $\phi : Y_i \to X$ of $\phi$ to each connected component $Y_i$ of $Y$ is an isomorphism.

b) A morphism $\phi : Y \to X$ of rigid spaces is called an *analytic* (*topological*) covering if there exists an admissible covering $\{X_i\}_{i\in I}$ of $X$ such that each covering $\phi^{-1}X_i \to X_i$ is a trivial covering.

c) A geometrically connected rigid space $X$ is called *simply connected* if every analytic covering of $X$ is trivial.

d) An analytic covering $\Omega \to X$ is called *the universal analytic covering of* $X$ if $\Omega$ is simply connected.

**Remark 3.41.** Choose a point $x_0 \in X$ and a point $\omega_0 \in \Omega$ satisfying $u(\omega_0) = x_0$. If $\Omega$ exists, then it satisfies the following universal property: If $\varphi : Y \to X$ is any connected (topological) covering, and $\varphi$ maps a fixed $y_0 \in Y$ to $x_0$, then there exists a unique (topological) covering $v : \Omega \to Y$ with $v(\omega_0) = y_0$ such that $\varphi \circ v = u$ ([FrPu04] Section 5.7(1)). The corresponding analytic morphism $u^{an} : \Omega^{an} \to X^{an}$ is also the universal covering for analytic spaces. This is true, because topological coverings of $X^{an}$ correspond to (rigid) topological covers of $X$ (See [Ber90] 3.3.4 or [deJ95] Proposition 5.3 and proof, see also the description below)

To construct the universal covering of a rigid curve $X$, it will be useful that $X$ has a suitable analytic reduction. The existence of such a reduction was remarked by Coleman [Co00] Section 1 (see also [FrPu04] Proposition 5.6.5 for an analytic description). We reproduce his remark with several comments and changes:

**Remark 3.42 (Coleman).** Suppose that $C$ is a smooth projective geometrically integral algebraic curve over $K$ with a minimal regular model $\mathcal{C}/V$ that is semi-stable (this can be assumed after a finite extension of $K$ by the semi-stable reduction theorem and [Liu02] Theorem 10.3.34). After a finite étale extension $V'$ of $V$ we can further assume that all singular points on the special fiber of $\mathcal{C}_{V'} := \mathcal{C} \times_{SpecV} SpecV'$ are split ordinary double points (apply [Liu02] Corollary 10.3.22 a) inductively to each singular point). Note that $\mathcal{C}_{V'}$ is still the minimal regular model (of $C \times_{SpecK} SpecK'$ for $K' = V'[1/p]$, over $V'$) and semi-stable by [Liu02] Corollary 10.3.36 a) and proof of c). We assume now that $C$ has already a minimal regular model $\mathcal{C}/V$ that is semi-stable and all singular points on the special fiber of $\mathcal{C}$ are split ordinary double points. Because $\mathcal{C}$ is regular and semi-stable, all singularities on the special fiber of $\mathcal{C}$ are of thickness one ([Liu02] Corollary 10.3.25 and $\mathcal{C}$ is already the minimal desingularization). After adjoining a square root of a uniformizer to $V$ we can assume that each singular point on the special fiber is of thickness 2. This is true, because if $x$ is a singular point on the special fiber, then there is an isomorphism $\hat{\mathcal{O}}_{\mathcal{C},x} \simeq V[[u,v]]/(uv-c)$ for some $c \in V$ with $v(c) = 1$ ([Liu02] Corollary 10.3.22 and $x$ is of thickness one), and adjoining a square root of a uniformizer to $V$ changes the valuation of $c$ to $v(c) = 2$. Then blowing up $\mathcal{C}$ once at all these singular points (of thickness 2) on the special fiber produces a regular semi-stable model and all irreducible components of the special fiber are non-singular. The problem is again local, so we fix a singular point $x$ on the special fiber of $\mathcal{C}$ and remove all other singular points from $\mathcal{C}$. Then we can use [Liu02] Lemma 10.3.21 and its proof. We then have to show that the blow up of $\mathrm{Spec}V[u,v]/(uv-c)$ ($c \in V$, $v(c) = 2$) at the point $(u, v, \pi)$ ($\pi$ a uniformizer) is regular and semi-stable with non-singular components on the special fiber. This was explicitly computed in [Liu02] Example 8.3.53.

The universal covering of a rigid curve $X/K$ is constructed in the following theorem:

**Theorem 3.43 (Existence of an universal covering).** - *Let $X$ be a geometrically irreducible, non-singular projective (rigid) curve over $K$. Then after replacing $K$ by a finite separable extension, $X$ has a universal analytic covering.*

*Proof.* [FrPu04] Theorem 5.7.2. For the convenience of the reader we recall the basic parts of the proof here: By Remark 3.42 we may suppose (after replacing $K$ by a finite extension) that there exists an analytic reduction $r : X \to Z$ with semi-stable $Z$ such that every irreducible component of $Z$ is a non-singular curve over $k$. Let $G$ be the intersection graph of $Z$. For each edge $e$ of $G$ one considers the affine open subset $U(e)$ of $Z$ obtained by removing all the irreducible components of $Z$ on which the double point $e$ does not lie. For each vertex $v$ of $G$ one defines the affine open subset $U(v)$ of $Z$ obtained by removing all irreducible components different from $v$. Then $\{r^{-1}U(v)\}_v \cup \{r^{-1}U(e)\}_e$ is a pure covering of $X$ which induces the analytic reduction $r : X \to Z$. Let $u_T : T \to G$ denote the universal covering of this graph. Then $T$ is a tree, locally isomorphic to $G$. For every edge $e$ of $T$ one defines $\Omega(e)$ to be the affinoid space $r^{-1}U(u_T e)$. For each vertex $v$ of $T$ one defines $\Omega(v)$ to be the affinoid space $r^{-1}U(u_T v)$. The space $\Omega$ is obtained by glueing the affinoid sets $\{\Omega(e)\}_e \cup \{\Omega(v)\}_v$ according to the tree $T$ □

We will now construct a formal scheme $\hat{\Omega}^\circ$. The construction is analogous to the construction in Theorem 3.43. The only difference is, that we will replace the affinoid algebras $A$ by $A^\circ$, the integral elements in $A$.

**Construction 3.44.** Let $C$ be a smooth geometrically integral projective algebraic curve over $K$. Assume that $C$ has a semi-stable (regular) model $\mathcal{C}$ such that its special fiber has non-singular components. Write $Z := \hat{\mathcal{C}}_s$ for the special fiber of the formal scheme $\hat{\mathcal{C}}$ associated to $\mathcal{C}$. By the assumptions $Z$ is a semi-stable curve over $K$ with non-singular components. Let $G$ be the intersection graph of $Z$. For each edge $e$ of $G$ one considers the affine open subset $U(e)$ of $Z$ obtained by removing all the irreducible components of $Z$ on which the double point $e$ does not lie. For each vertex $v$ of $G$ one defines the affine open subset $U(v)$ of $Z$ obtained by removing all irreducible components different from $v$. Let $u_T : T \to G$ denote the universal covering of the graph $G$. Then $T$ is a tree, locally isomorphic to $G$. For every edge $e$ of $T$ one defines $\hat{\Omega}^\circ(e)$ to be the affine formal scheme $\mathrm{Spf}\mathcal{O}_{\hat{\mathcal{C}}}(U(u_T e))$. For each vertex $v$ of $T$ one defines $\hat{\Omega}^\circ(v)$ to be the affine formal scheme $\mathrm{Spf}\mathcal{O}_{\hat{\mathcal{C}}}(U(u_T v))$. The formal scheme $\hat{\Omega}^\circ$ is obtained by glueing the affine formal schemes $\{\hat{\Omega}^\circ(e)\}_e \cup \{\hat{\Omega}^\circ(v)\}_v$ according to the tree $T$. In this way one obtains a topological covering $u : \hat{\Omega}^\circ \to \hat{\mathcal{C}}$ in the Zariski topology.

**Remark 3.45.** a) We assume that one can show that $\hat{\Omega}^\circ$ is the universal covering of $\hat{\mathcal{C}}$ in the Zariski topology. The proof should be analogous to [FrPu04] Section 5.7. We will not need this in this paper.

b) The formal scheme $\hat{\Omega}^\circ$ is admissible. This is true, because locally it is isomorphic to $\hat{\mathcal{C}}$ and we claim that this formal scheme is admissible. Again this question is local, so let $\mathrm{Spec}A \subset \mathcal{C}$ be an affine open, and denote by $\hat{A}$ its formal completion. Then $\hat{A}$ is of topological finite presentation over $\mathcal{O}_K$ and also flat, because $A$ is flat over $\mathcal{O}_K$ and $\hat{A}$ is flat over $A$ (Lemma 3.35). The flatness of $\hat{A}$ over $\mathcal{O}_K$ implies that $\hat{A}$ is admissible ([BoLu93] Remarks before Proposition 1.1)

c) The admissible formal scheme $\hat{\Omega}^\circ$ has a rigid generic fiber ([BoLu93] Section 4). If $\mathrm{Spf}A \subset \hat{\Omega}^\circ$ is an admissible open affine $\mathcal{O}_K$-algebra, then $A_{rig} := A \otimes_{\mathcal{O}_K} K$ is an affinoid $K$-algebra. The generic fiber of $\hat{\Omega}^\circ$ is obtained by glueing these affinoid algebras [BoLu93] Section 4. From the construction in $\Omega$ (Theorem 3.43) and $\hat{\Omega}^\circ$ we can see that $\Omega$ is the generic fiber of $\hat{\Omega}^\circ$.

d) A deck-transformation of $u : \hat{\Omega}^\circ \to \hat{\mathcal{C}}$ gives rise to a deck-transformation of $u : \Omega \to C^{rig}$ and vice versa. The group of deck-transformations of $u : \Omega \to C^{rig}$ is isomorphic to the topological fundamental group of the graph $G$ corresponding to $Z$ [FrPu04] proof of Theorem 5.7.2.

e) By construction of the universal covering, the group of deck-transformations $\Gamma$ acts freely and discountinuously in the Zariski topology of the special fiber of $\hat{\Omega}^\circ$.

f) If we replace $C$ by $C_{\mathbb{C}_p}$ and $\mathcal{C}$ by $\mathcal{C}_{\mathfrak{o}}$ (i.e. their base change with $\mathbb{C}_p$ and $\mathfrak{o}$) then we can perform the previous construction also over $\mathfrak{o}$ or $\mathbb{C}_p$. All points stated above, Construction 3.44 and Theorem 3.43 are valid over $\mathfrak{o}$ and $\mathbb{C}_p$ as well.

The following corollary is already known in the case of Mumford curves [Her05] Chapter 1 Corollary 1.48

**Corollary 3.46.** *Let $C/K$ be a smooth geometrically integral projective curve and assume that $C$ admits a semi-stable $V$-model $\mathcal{C}$ with non-singular components on the special fiber. Let $L \subset \mathbb{C}_p$ be a complete subfield. Then the rigid fundamental groups of $C$ and $C_L$ are isomorphic.*

*Proof.* The models $\mathcal{C}$ and $\mathcal{C} \times_{\mathrm{Spec}V} \mathrm{Spec}\mathcal{O}_L$ have the same reduction graph because $\mathcal{C}$ is semi-stable. The rigid fundamental group of $C$ resp. $C_L$ is isomorphic to the reduction graph of $\mathcal{C}$ resp. $\mathcal{C} \times_{\mathrm{Spec}V} \mathrm{Spec}\mathcal{O}_L$ □

## 3.9 A $p$-adic Riemann-Hilbert correspondence

| | |
|---|---|
| $K$ | a complete non-archimedean field |
| $V$ | its ring of integers |
| $S$ | a connected $K$-manifold |
| $\overline{s}$ | a geometric point of $S$ |

On a complex manifold one can attach to every finite-dimensional representation of the fundamental group a vector-bundle with integrable connection, and vice versa. This is the so called *Riemann-Hilbert correspondence.*

In [And03] Yves André defined a $p$-adic analogue on Berkovich-spaces. We present his results here.

**Proposition 3.47 (Y. André).** - *There is a natural equivalence of categories*

$$\{discrete\ \pi_1^{et}(S,\overline{s}) - representations\} \cong \{\acute{e}tale\ local\ systems\ on\ S\}.$$

*(Here representations are representation on the automorphism group of finite dimensional $K$-vector spaces).*

*Proof.* [And03] Y.André Proposition III.3.4.4 □

**Theorem 3.48 (Y. André).** - *There is a natural equivalence of Tannakian categories (the* non-archimedean étale Riemann-Hilbert functor*):*

$$\{discrete\ \pi_1^{et}(S,\overline{s}) - representations\ (on\ fin.\ dim.\ K\text{-}vector\ spaces)\}$$

$$\xrightarrow{RH^{et}}$$

$$\{vector\ bundles\ with\ integrable\ connection\ (M,\nabla)\ on\ S$$
$$such\ that\ M_{et}^{\nabla}\ is\ an\ \acute{e}tale\ local\ system\}.$$

*The correspondence is given by*

$$\mathcal{V}_\rho \longmapsto (M_\rho, \nabla_\rho) := (\mathcal{V} \otimes_K \mathcal{O}_{S^{et}}, id \otimes d_{S^{et}})$$

$$(M,\nabla) \longleftarrow M_{et}^{\nabla}$$

*where $\mathcal{V}_\rho$ is the local system corresponding to a finite dimensional representation $\rho$, and $M_{et}^{\nabla}$ is the sheaf of horizontal sections. In this correspondence, the subspace of $\pi_1^{et}(S,\overline{s})$-invariants corresponds to the space of global sections of $M_{et}^{\nabla}$.*

*Proof.* Y. Andre [And03] Theorem III 3.4.6. □

**Remark 3.49.** The main open problem is to describe the image of the Riemann Hilbert functor, i.e. those vector bundles with integrable connection $(M,\nabla)$ on $S$, such that $M_{et}^{\nabla}$ is an étale local system.

We are interested in some special cases of Andres' Riemann Hilbert correspondence: For any full subcategory $Cov_S^{\bullet} \subset Cov_S^{et}$ which is stable under taking connected components, fiber products and quotients, there is a natural continuous homomorphism:

$$\pi_1^{et}(S,\overline{s}) \to \pi_1^{\bullet}(S,\overline{s})$$

with dense image ([And03] Corollary 1.4.8). Any discrete representation of $\pi_1^\bullet(S,\overline{s})$ gives rise to a discrete representation of $\pi_1^{et}(S,\overline{s})$ with the same co-image. It follows, that the étale Riemann-Hilbert functor induces a fully faithful functor

$$\{\text{discrete } \pi_1^\bullet(S,\overline{s})\text{-representations on fin. dim. } K\text{-vector spaces}\}$$

$$\xrightarrow{RH^\bullet}$$

$$\{\text{vector bundles with integrable connection } (M,\nabla) \text{ on } S\}$$

(Y. André [And03] III Example 3.5.1). Of special interest for us is the case that $\bullet = top$ i.e. we are interested in the topological fundamental group $\pi_1^{top}(S,\overline{s})$, and the topological Riemann-Hilbert functor $RH^{top}$ (See also [And03] I 1.5).

There are some results describing the image of the topological Riemann-Hilbert functor:

**Theorem 3.50 (Faltings, van der Put, Reversat).** - *Let $X$ be a Mumford-curve over a closed subfield $K$ of $\mathbb{C}_p$, and $\overline{x}$ a geometric base point.*

a) *If $\rho$ is a $\Phi$-bounded representation of $\Gamma := \pi_1^{top}(X_{\mathbb{C}_p},\overline{x})$ into $GL_r(K)$, then $M_\rho$ is semi-stable of degree zero.*

b) *For any semi-stable vector bundle $M$ on $X$ of degree zero there exists a $\Phi$-bounded representation $\rho$ with $M \cong M_\rho$*

b) *If $X = K^*/q^{\mathbb{Z}}$, $v(q) = m > 0$ is a Tate-curve, then the absolutely indecomposable representations of $\Gamma$ are obtained by sending a generator of $\Gamma = \mathbb{Z}$ to*

$$\begin{pmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}$$

*with $\lambda \in K^*$. Such a representation is $\Phi$-bounded, if and only if $0 \leq v(\lambda) < m$ ($v$ is the $p$-adic valuation on $K$).*

*Proof.* [Fal83], [PuRe86] □

**Remark 3.51.** -

a) $\Phi$-*bounded representations* are representations whose coefficients satisfy certain growth conditions. Integral representations are $\Phi$-bounded. In this correspondence connections do not appear.

b) If $S$ is an analytic torus then homogeneous (translation invariant) vector bundles on $S$ are in one to one correspondence with $\Phi$-bounded representations of the topological fundamental group of $S$ [PuRe88].

c) It was shown by Florentino [Flo01] Theorem 1 that all *maximally unstable* vector bundles on a *Schottky uniformized* Riemann surface are induced by representations of the Schottky group. A similar result should hold for maximally unstable vector bundles on Mumford curves, where the topological fundamental group is also a Schottky group.

Let $K$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $V$. Let $X$ be a smooth geometrically integral projective curve over $K$ admitting a semistable (regular) model with non-singular components on the special fiber as in 3.8. We will present the construction of Y. Andrés' topological Riemann-Hilbert functor $RH^{top}$ in detail in the case of curves. We will follow closely the construction of Reversat and van der Put in [PuRe86] (1.11)-(1.13) and Faltings [Fal83], Gieseker [Gie73]. Their construction can be extended to curves with arbitrary reduction, to integral representations and integral vector bundles, and to vector bundles with connections:

**Construction 3.52.** - Let $\mathcal{X}^{nsc}$ be a $V$-model of $X$ as constructed in Section 3.8 and let $\hat{\mathcal{X}}^{nsc}$ be the formal completion of $\mathcal{X}^{nsc}$ for the ideal defined by $(p)$. Let $u : \hat{\Omega}^\circ \to \hat{\mathcal{X}}^{nsc}$ be its universal covering. Denote by $\hat{\mathcal{X}}^{nsc}_s$, $\hat{\Omega}^\circ_s$ their special fibers. Denote by $\Gamma$ the automorphism group of $\hat{\Omega}^\circ/\hat{\mathcal{X}}^{nsc}$, and let $\rho : \Gamma \to Aut_{\mathfrak{o}}(\mathbb{L})$ be an integral representation on a free $\mathfrak{o}$-module $\mathbb{L}$ of rank $r$. Set $\hat{\mathcal{X}}^{nsc}_{\mathfrak{o}} = \hat{\mathcal{X}}^{nsc} \otimes_V \mathfrak{o}$ for the base change with $\mathfrak{o}$, and similar for $\hat{\Omega}^\circ$. We define a $\Gamma$-action on $\mathcal{O}_{\hat{\Omega}^\circ}$ by the rule $\gamma(f) := (\gamma^{-1})^* f$ (This is consistent with the literature [Gro56] Section 2 or [PuRe86]).

a) For $U \subset \hat{\mathcal{X}}^{nsc}_{\mathfrak{o},s}$ open we set

$$M^\circ_\rho(U) := \{m \in \mathbb{L} \otimes_{\mathfrak{o}} \mathcal{O}_{\hat{\Omega}^\circ_{\mathfrak{o}}}(u^{-1}U) | \gamma(m) = m \text{ for all } \gamma \in \Gamma\}.$$

Here the action of $\Gamma$ on $(\mathbb{L} \otimes_{\mathfrak{o}} \mathcal{O}_{\hat{\Omega}^\circ_{\mathfrak{o}}}(u^{-1}U))$ is the diagonal action, i.e.

$$\gamma(\sum_{i=1}^r e_i \otimes f_i) = \sum_{i=1}^r \rho(\gamma)(e_i) \otimes \gamma(f_i) \text{ for } e_i \in \mathbb{L},\ f_i \in \mathcal{O}_{\hat{\Omega}^\circ_{\mathfrak{o}}}(u^{-1}U).$$

We claim that this defines a formal vector bundle on $\hat{\mathcal{X}}^{nsc}_{\mathfrak{o}}$.

b) We can also define a connection $\nabla^\circ_\rho$ on $M^\circ_\rho$ by setting

$$\nabla^\circ_\rho(\sum_{i=1}^r e_i \otimes f_i) := \sum_{i=1}^r e_i \otimes df_i$$

for $U \subset \hat{\mathcal{X}}_{\mathfrak{o},s}^{nsc}$ open, and $e_i \in \mathbb{L}$, $f_i \in \mathcal{O}_{\hat{\Omega}_{\mathfrak{o}}^\circ}(u^{-1}U)$. This map is $\Gamma$-equivariant:

$$\gamma(\nabla_\rho^\circ(\sum_{i=1}^r e_i \otimes f_i)) = \sum_{i=1}^r \rho(\gamma)(e_i) \otimes \gamma(df_i) =$$

$$\sum_{i=1}^r \rho(\gamma)(e_i) \otimes d\gamma(f_i) = \nabla_\rho^\circ(\gamma(\sum_{i=1}^r e_i \otimes f_i)).$$

We claim that it descends to a connection $\nabla_\rho : M_\rho^\circ \to M_\rho^\circ \otimes_{\mathcal{O}_{\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}}} \hat{\Omega}^1_{\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}/\mathfrak{o}}$. Here the sheaf of formal differential forms is defined to be the projective limit of the sheafs of algebraic differential forms modulo $p^n$. The sheaf of formal differential forms $\hat{\Omega}^1_{\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}/\mathfrak{o}}$ is thus isomorphic to

$$\lim_{\leftarrow} \Omega^1_{(\mathcal{X}_{\mathfrak{o}}^{nsc})_n/\mathfrak{o}_n} \cong \lim_{\leftarrow} \Omega^1_{\mathcal{X}_{\mathfrak{o}}^{nsc}/\mathfrak{o}} \otimes_{\mathfrak{o}} \mathfrak{o}_n.$$

For more details about the module of formal differential forms we refer the reader to [EGA] IV (Premiere Partie) Section 20.7.

**Remark 3.53.** It was already mentioned in [Gie73] Lemma 2 that a coherent sheaf with descent datum descends to a coherent sheaf in the case that $X$ is a Mumford curve and the formal scheme is noetherian. Descent theory is also available in our situation (over $\mathfrak{o}$) [BoGo98] Theorem 2.1. We will calculate the sheaf of invariant sections and the connection explicitly, so we need not to make use of the descent theory mentioned above.

**Proposition 3.54.** *The sheaf $M_\rho^\circ$ defined in Construction 3.52 a) is a vector bundle (a locally free sheaf of rank $r$) on $\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}$. The connection defined in Construction 3.52 b) descends to a connection on $M_\rho^\circ$.*

*Proof.* The assertion is local so we can take a small Zariski open $U \subset \hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}$ such that $u^{-1}U = \coprod_{g\in\Gamma} U$, i.e. the inverse image of $U$ in $\hat{\Omega}^\circ$ is a disjoint union of copies of $U$ indexed by $\Gamma$. This is possible because $\hat{\Omega}_{\mathfrak{o}}^\circ$ is a covering of $\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}$ with deck-transformation group $\Gamma$ in the Zariski topology. But then we are in the situation of Lemma 3.55 with $A := \mathcal{O}_{\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}}(U)$, $B := \mathcal{O}_{\hat{\Omega}_{\mathfrak{o}}^{nsc}}(u^{-1}U)$, $W := \mathbb{L}$, $R = \mathfrak{o}$ and $G = \Gamma$ and so $M := M_\rho^\circ(U)$ is a free $A = \mathcal{O}_{\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}}(U)$-module of rank $r$. To show that the connection descends we can work modulo $p^n$. We can then set $C := \Omega^1_{(\mathcal{X}_{\mathfrak{o}}^{nsc})_n/\mathfrak{o}_n}(U)$. If we replace $B$, $A$, $W$, $R$ by their reductions modulo $p$ then we are in the situation of Lemma 3.56 and the connection descends modulo $p^n$. Because of the explicit description in Equation 1 we see that we get a projective system of such maps for varying $n$, hence a map on the projective limit. So $\nabla_\rho$ descends $\square$

**Lemma 3.55.** *Let $R$ be a commutative ring with unit, and let $A$ be a commutative $R$-algebra with unit. Let $G$ be a group with a simply transitive action on the set $H := G$. Let $B := \prod_{h\in H} A_h$ be a direct product of copies of $A$, i.e. $A_h := A$. Define a $G$-action on $B$ by the rule $g((a_h)) = (a_{g(h)})$, i.e. $G$ permutes the ordering of an element. Then $B$ is an $A$ algebra and the ring of $G$-invariants $B^G$ can be identified with $A$. Let $W := e_1R \oplus \ldots \oplus e_rR$ be a free $R$-module of rank $r$ with basis $e_1, \ldots, e_r$. Let $\rho : G \to GL_r(R)$ be a representation. Define a semi-linear $G$-action on $M := W \otimes_R B$ by the rule*

$$g(w \otimes_R b) := \rho(g) \cdot w \otimes_R g(b) \qquad w \in W,\ b \in B$$

*Then the $A$-module $M^G$ of $G$-invariant elements is free of rank $r$.*

*Proof.* An element $(a_h) \in B$ is $G$-invariant if and only if $a_{g(h)} = a_h$ for all $g \in G$ and all $h \in H$. As $G$ acts simply transitive on $H$ this is the case if and only if $a_{h'} = a_h$ for all $h, h' \in H$, so $(a_h)$ is represented by a single element $a \in A$ and so $B^G = A$. We will give now an explicit description of $M^G$: An element $m \in M$ can be written as

$$m = \sum_{k=1}^{r} e_k \otimes (a_h)_k \qquad \text{for } (a_h)_k \in B$$

(For simplicity we have written the coefficients in $R$ belonging to the $e_k$ on the right hand side). By identifying $M$ with $B^r$ we can write $m$ as a column vector, i.e.

$$m = \begin{pmatrix} (a_h)_1 \\ \vdots \\ (a_h)_r \end{pmatrix}.$$

The $G$-action on $m$ is given by the rule

$$g(m) = g \begin{pmatrix} (a_h)_1 \\ \vdots \\ (a_h)_r \end{pmatrix} = \rho(g) \cdot \begin{pmatrix} g((a_h)_1) \\ \vdots \\ g((a_h)_r) \end{pmatrix}$$

if $\rho(g)$ is considered as a matrix with coefficients in $R \otimes_R B \cong B$. An element $m \in M$ is invariant if and only if

$$\rho(g) \cdot \begin{pmatrix} g((a_h)_1) \\ \vdots \\ g((a_h)_r) \end{pmatrix} = \begin{pmatrix} (a_h)_1 \\ \vdots \\ (a_h)_r \end{pmatrix}$$

for all $g \in G$. This is the case if and only if

$$\rho(g) \cdot \begin{pmatrix} a_{g(h),1} \\ \vdots \\ a_{g(h),r} \end{pmatrix} = \begin{pmatrix} a_{h,1} \\ \vdots \\ a_{h,r} \end{pmatrix}$$

for all $g \in G$ and $h \in H$. Because $G$ acts simply transitive on $H$ we can deduce, that an element $m \in M$ is fixed under the $G$-action if and only if there are elements $a_1, \dots, a_r \in A$ such that $m = ((a_h)_1, \dots, (a_h)_r)^T$ with $a_{h^0,i} = a_i$ for $i = 1, \dots, r$ and some fixed $h^0$, and $a_{g(h^0),i}$ defined for $i = 1, \dots, r$ by the rule

$$\begin{pmatrix} a_{g(h^0),1} \\ \vdots \\ a_{g(h^0),r} \end{pmatrix} = \rho(g)^{-1} \cdot \begin{pmatrix} a_{h^0,1} \\ \vdots \\ a_{h^0,r} \end{pmatrix}$$

This shows that $M^G$ is a free $A$-module of rank $r$ □

**Lemma 3.56.** *We use the same notation as in Lemma 3.55. Denote by $C := \Omega^1_{A/R}$ the module of differential forms. Set $D := \prod_{g \in G} C_h$ for the direct product of copies $C_h := C$ indexed by the set $H$. Let $G$ act on $D$ by the translation map, i.e. $g((c_h)) := (c_{g(h)})$. Define a map*

$$\nabla : M \to M \otimes_B D \qquad w \otimes_R b \mapsto w \otimes_R 1 \otimes_B db$$

*($w \in W$, $b \in B$). Then $\nabla$ descends to a map of the fixed modules, i.e. $\nabla(M^G) \subset M^G \otimes_A D^G = M^G \otimes_A C$*

*Proof.* As in the proof of Lemma 3.55 an $G$-invariant element of $M$ can be written as $m = ((a_h)_1, \dots, (a_h)_r)^T$ with $a_{h^0,i} = a_i$ for $i = 1, \dots, r$ and some fixed $h^0$ for $a_1, \dots, a_r \in A$, and $a_{g(h^0),i}$ defined for $i = 1, \dots, r$ by the rule

$$\begin{pmatrix} a_{g(h^0),1} \\ \vdots \\ a_{g(h^0),r} \end{pmatrix} = \rho(g)^{-1} \cdot \begin{pmatrix} a_{h^0,1} \\ \vdots \\ a_{h^0,r} \end{pmatrix}.$$

By definition $\nabla$ maps $m = ((a_h)_1, \dots, (a_h)_r)^T$ to $((da_h)_1, \dots, (da_h)_r)^T$. Define elements $(x_h)^1, \dots, (x_h)^r \in M = W \otimes_R B$ by the rule

$$\begin{pmatrix} x^i_{g(h^0),1} \\ \vdots \\ x^i_{g(h^0),r} \end{pmatrix} = \rho(g)^{-1} \cdot e_i$$

for $i = 1, \dots, r$ and some fixed $h^0$ (Note that by definition these elements are in $M^G$). Then

$$\begin{pmatrix} da_{g(h^0),1} \\ \vdots \\ da_{g(h^0),r} \end{pmatrix} = \rho(g)^{-1} \cdot \begin{pmatrix} da_{h^0,1} \\ \vdots \\ da_{h^0,r} \end{pmatrix} = x^1_{g(h^0)} \otimes_B da_{h^0,1} + \ldots + x^r_{g(h^0)} \otimes_B da_{h^0,r}.$$

for all $g \in G$. Hence

$$\nabla(m) = (x_h)^1 \otimes_B da_{h^0,1} + \ldots + (x_h)^r \otimes_B da_{h^0,1r} \in M^G \otimes_A C \qquad (1)$$

because $(x_h)^i \in M^G$ and $a_{h^0,i} \in A$ for all $i = 1, \dots, r$ □

**Remark 3.57.** The rigid generic fiber $M_\rho^{rig} := M_\rho^\circ \otimes_{\mathfrak{o}} \mathbb{C}_p$ of $M_\rho^\circ$ and the connection $\nabla_\rho^\circ \otimes_{\mathfrak{o}} \mathbb{C}_p$ coincide with the rigid vector bundle defined in [PuRe86]. This can be seen locally. If $Red : X^{rig}_{\mathbb{C}_p} \to \mathcal{X}^{nsc}_{\mathfrak{o},s}$ denotes the reduction map corresponding to the model $\mathcal{X}^{nsc}_{\mathfrak{o}}$, then for $U \subset \mathcal{X}^{nsc}_{\mathfrak{o}}$ affine open, $Red^{-1}U$ is an admissible affinoid subset of $X^{rig}_{\mathbb{C}_p}$. Then

$$(M_\rho^\circ \otimes_{\mathfrak{o}} \mathbb{C}_p)(U) = \{m \in \mathbb{L} \otimes_{\mathfrak{o}} \mathcal{O}_{\hat{\Omega}^\circ_{\mathfrak{o}}}(u^{-1}U) | \gamma(m) = m \text{ for all } \gamma \in \Gamma\} \otimes_{\mathfrak{o}} \mathbb{C}_p =$$

$$= \{m \in \mathbb{L}_{\mathbb{C}_p} \otimes_{\mathbb{C}_p} \mathcal{O}_{\Omega_{\mathbb{C}_p}}(u^{-1}Red^{-1}U) | \gamma(m) = m \text{ for all } \gamma \in \Gamma\} = M_\rho^{rig}.$$

And similar for the connection. From the calculation one sees that $(M_\rho^{rig}, \nabla_\rho)$ does not depend on the chosen model $\mathcal{X}^{nsc}$. The vector bundle with connection $(M_\rho, \nabla)$ on $X^{an}$ defined in Theorem 3.48 corresponds to the rigid vector bundle with connection $(M_\rho^{rig}, \nabla_\rho)$ (We refer the reader to [Ber93] Section 1.6 for the correspondence between vector bundles on analytic and rigid spaces). We will write $RH^{top,\circ}$ for the *integral* topological Riemann-Hilbert functor defined above, i.e. $RH^{top,\circ}(\rho) := (M_\rho^\circ, \nabla_\rho^\circ)$.

## 3.10 Berkovich $p$-adic integration

$K$ a non-Archimedean field in characteristic 0
$X$ a smooth $K$-analytic space

We recall the main results of Berkovichs' book [Ber07] on $p$-adic integration:

**Remark 3.58.** We use $K$ for the base field, whereas Berkovich uses $k$. We restrict for simplicity to characteristic 0.

**Definition 3.59 (Berkovich).** -

a) The sheaf of *constant analytic functions* is defined as $\mathfrak{c}_X = Ker(\mathcal{O}_X \xrightarrow{d} \Omega^1_X)$.

b) Let $\mathbf{K}$ be a *filtered* $K$ algebra, i.e. a commutative $K$-algebra with unity provided with an increasing sequence of $K$-vector subspaces $\mathbf{K}^0 \subset \mathbf{K}^1 \subset \ldots$ such that $\mathbf{K}^i \cdot \mathbf{K}^j \subset \mathbf{K}^{i+j}$ and $\mathbf{K} = \cup_{i=0}^{\infty} \mathbf{K}^i$. Given a strictly $K$-analytic space $X$, we set $\mathcal{O}_X^{\mathbf{K},i} = \mathcal{O} \otimes_K \mathbf{K}^i$. If $X$ is reduced, we set $\mathcal{C}_X^{\mathbf{K},i} = \mathfrak{c}_X \otimes_K \mathbf{K}^i$.

**Definition 3.60 (Berkovich).** -

a) Let $X$ be a smooth $K$-analytic space. A $\mathcal{D}_X$-*module* on $X$ is an étale $\mathcal{O}_X$-module $\mathcal{F}$ provided with an integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1$.

b) A $\mathcal{D}_X$-*algebra* is an étale commutative $\mathcal{O}_X$-algebra $\mathcal{A}$ which is also a $\mathcal{D}_X$-module whose connection $\nabla$ satisfies the Leibnitz rule $\nabla(g \cdot f) = fdg + gdf$. If in addition $\mathcal{A}$ is a filtered $\mathcal{O}_X$-algebra such that all $\mathcal{A}^i$ are $\mathcal{D}_X$-submodules of $\mathcal{A}$, then $\mathcal{A}$ is said to be a *filtered* $\mathcal{D}_X$-*algebra*.

**Theorem 3.61 (Berkovich).** *Given a closed subfield $K \subset \mathbb{C}_p$, a filtered $K$-algebra $\mathbf{K}$ and an element $\lambda \in \mathbf{K}^1$, there is an unique way to provide every smooth $K$-analytic space $X$ with a filtered $\mathcal{D}_X$-algebra $\mathcal{S}_X^\lambda$ such that the following is true:*

*a)* $\mathcal{S}_X^{\lambda,0} = \mathcal{O}_X^{\mathbf{K},0}$;

*b)* $Ker(\mathcal{S}_X^{\lambda,i} \xrightarrow{d} \Omega^1_{\mathcal{S}^{\lambda,i},X}) = \mathcal{C}_X^{K,i}$;

*c)* $Ker(\Omega^1_{\mathcal{S}^{\lambda,i},X} \xrightarrow{d} \Omega^2_{\mathcal{S}^{\lambda,i},X}) \subset d\mathcal{S}_X^{\lambda,i+1}$;

*d)* $\mathcal{S}_X^{\lambda,i+1}$ *is generated by local sections $f$ for which $df$ is a local section of* $Ker(\Omega^1_{\mathcal{S}^{\lambda,i},X} \xrightarrow{d} \Omega^2_{\mathcal{S}^{\lambda,i},X})$;

*e)* $Log^\lambda(T) \in \mathcal{S}^{\lambda,1}(\mathbb{G}_m)$.

*f) for any morphism of smooth $K$-analytic spaces $\phi : X' \to X$, one has* $\phi^*(\mathcal{S}_X^{\lambda,i}) \subset \mathcal{S}_{X'}^{\lambda,i}$.

*Proof.* [Ber07] Theorem 1.6.1 □

**Definition 3.62 (Berkovich).** - A $\mathcal{D}_X$-module $\mathcal{F}$ on a smooth $K$-analytic space $X$ is said to be *unipotent* if there exists a sequence of $\mathcal{D}_X$-submodules $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \ldots \subset \mathcal{F}^n = \mathcal{F}$ such that all of the quotients $\mathcal{F}^i/\mathcal{F}^{i-1}$ are isomorphic to the trivial $\mathcal{D}_X$-module $\mathcal{O}_X$. A $\mathcal{D}_X$-module $\mathcal{F}$ is said to be *unipotent* (resp. *quasi-unipotent*) at a point $x \in X$, if $x$ has an open neighborhood $U \subset X$ (resp. an étale neighborhood $U \to X$) for which $\mathcal{F}|_U$

is unipotent. A $\mathcal{D}_X$-module $\mathcal{F}$ is said to be *locally unipotent* (resp. *quasi-unipotent*) if it is unipotent (resp. quasi-unipotent) at all points of $X$.

Furthermore the *level* of a unipotent $\mathcal{D}_X$-module $\mathcal{F}$ on $X$ is the minimal $n$ for which there is a filtration of $\mathcal{D}_X$-submodules $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \ldots \subset \mathcal{F}^n = \mathcal{F}$ such that each quotient $\mathcal{F}^i/\mathcal{F}^{i-1}$ is a trivial $\mathcal{D}_X$-module. If a $\mathcal{D}_X$-module $\mathcal{F}$ is unipotent (resp. quasi-unipotent) at a point $x \in X$, its level at $x$ is the minimal number $n$, which is the level of the unipotent $\mathcal{D}_U$-module $\mathcal{F}|_U$ for some $U$ (from the previous paragraph).

**Lemma 3.63.** *Let $\mathcal{F}$ be a $\mathcal{O}_X$-coherent $\mathcal{D}_X$ - module, $x \in X$ and $n \geq 1$. Then the following statements are equivalent:*

*a) $\mathcal{F}$ is quasi-unipotent at $x$ of level at most $n$;*

*b) the point $x$ has an étale neighborhood $U \to X$ such that, for some $m \geq 1$, there is an embedding of $\mathcal{D}_U$-modules $\mathcal{F}|_U \hookrightarrow (\mathcal{S}_U^{\lambda,n-1})^m$.*

*Proof.* [Ber07] Theorem 9.3.3 □

**Theorem 3.64 (Berkovich).** *There is a unique way to construct, for every closed subfield $K \subset \mathbb{C}_p$, every filtered $K$-algebra $\mathbf{K}$, every element $\lambda \in \mathbf{K}^1$, every connected smooth $K$-analytic space $X$ with $\pi_1^{top}(\overline{X}) \tilde{\to} \pi_1^{top}(X)$, every locally unipotent $\mathcal{D}$-module $\mathcal{F}$ on $X$ and every path $\gamma : [0,1] \to X$ with ends $x, y \in X(K)$ (also $x, y \in X_{st,K}$ [Ber07] Page 3), an isomorphism (*parallel transport*) of $\mathbf{K}$-modules*

$$T_\gamma^{\mathcal{F}} = T_\gamma^{\mathcal{F},\lambda} : \mathcal{F}_x^\nabla \otimes_K \mathbf{K} \tilde{\to} \mathcal{F}_y^\nabla \otimes_K \mathbf{K}$$

*such that the following is true:*

*a) $T_\gamma^{\mathcal{F}}$ depends only on the homotopy type of $\gamma$;*

*b) given a second path $\tau : [0,1] \to X$ with ends $y, z \in X_{st,K}$, one has $T_{\tau\circ\gamma}^{\mathcal{F}} = T_\tau^{\mathcal{F}} \circ T_\gamma^{\mathcal{F}}$;*

*c) if $\mathcal{F}$ is the unipotent $\mathcal{D}$-module $\mathcal{O}_X e_1 \oplus \mathcal{O}_X e_2$ on $X = \mathbb{G}_m$ with $\nabla(e_1) = 0$ and $\nabla(e_2) = \frac{dT}{T} e_1$, $\gamma(0) = 1$ and $\gamma(1) = a \in K^*$, then $T_\gamma^{\mathcal{F}}(e_2 - log(T)e_1) = (e_2 - log(\frac{T}{a})e_1) - Log^\lambda(a)e_1$;*

*d) $T_\gamma^{\mathcal{F}}$ is functorial with respect to $\mathcal{F}$;*

*e) $T_\gamma^{\mathcal{F}}$ commutes with tensor products;*

*f) $T_\gamma^{\mathcal{F}}$ is functorial with respect to $(K, X, \gamma, \mathbf{K}, \lambda)$.*

*Furthermore, the parallel transport posses the following properties:*

*1) $T_\gamma^{\mathcal{F}}$ commutes with the Hom-functor;*

*2) If $\mathcal{F}$ is unipotent of level $n$, then $T_\gamma^{\mathcal{F}}(\mathcal{F}_x^\nabla) \subset \mathcal{F}_y^\nabla \otimes_K \mathbf{K}^{n-1}$;*

*3) If $\mathcal{F}$ is unipotent and $\gamma([0,1]) \subset Y$, where $Y$ is an analytic domain with good reduction, then $T_\gamma^{\mathcal{F}}(\mathcal{F}_x^\nabla) \subset \mathcal{F}_x^\nabla$.*

*Proof.* [Ber07] Theorem 9.4.1. The proof is interesting for our purposes, so we will reproduce it here: Assume first, that $K = \mathbb{C}_p$. In this case the condition on $X$ is automatically satisfied, $\mathfrak{c}_X$ is the constant sheaf $K_X$ associated to $K$, and $X_{st,K} = X_{st}$. From Lemma 3.63 it follows that the sheaf $\gamma^*(\mathcal{F}_{\mathcal{S}^\lambda}^\nabla)$ is constant. It follows, that there are canonical isomorphisms from $\gamma^*(\mathcal{F}_{\mathcal{S}^\lambda}^\nabla)([0,1])$ onto $\gamma^*(\mathcal{F}_{\mathcal{S}^\lambda}^\nabla)_0 = \mathcal{F}_{\gamma(0)}^\nabla \otimes_K \mathbf{K}$ and $\gamma^*(\mathcal{F}_{\mathcal{S}^\lambda}^\nabla)_1 = \mathcal{F}_{\gamma(1)}^\nabla \otimes_K \mathbf{K}$, and so they give rise to an isomorphism of $\mathbf{K}$-modules

$$T_\gamma^{\mathcal{F}} : \mathcal{F}_{\gamma(1)}^\nabla \otimes_K \mathbf{K} \tilde{\to} \mathcal{F}_{\gamma(1)}^\nabla \otimes_K \mathbf{K}.$$

If $K$ is not necessarily algebraically closed, we construct the parallel transport as follows: By ([Ber07] Lemma 9.1.2), there exists a path $\gamma' : [0,1] \to \overline{X}$ with $\alpha \circ \gamma' = \gamma$. Since the points $x = \gamma(0)$ and $y = \gamma(1)$ are in $X_{st,K}$, they have unique pre-images $x'$ and $y'$ in $\overline{X}$, respectively, and it follows that $\gamma'(0) = x'$ and $\gamma'(1) = y'$. We denote by $\overline{\mathcal{F}}$ the pullback $\mathcal{F}$ on $\overline{X}$ and, for an element $f \in \mathcal{F}_x^\nabla$ we set

$$T_\gamma^{\mathcal{F}}(f) = T_{\gamma'}^{\overline{\mathcal{F}}}(f) \in \mathcal{F}_{y'}^\nabla \otimes_{\mathbb{C}_p} (\mathbf{K} \otimes_K \mathbb{C}_p) = (\mathcal{F}_y^\nabla \otimes_K \mathbf{K}) \otimes_K \mathbb{C}_p.$$

First of all, the element $T_{\gamma'}^{\overline{\mathcal{F}}}(f)$ does not depend on the choice of $\gamma'$. Indeed, if $\gamma''$ is another lifting of $\gamma$, then the loop $\gamma'^{-1} \circ \gamma''$ is homotopy trivial (since $\pi_1(\overline{X}) \tilde{\to} \pi_1(X)$ ) and, therefore $T_{\gamma''}^{\overline{\mathcal{F}}}(f) = T_{\gamma'}^{\overline{\mathcal{F}}}(f)$. Furthermore given an element $\sigma$ of the Galois group $G$ of $\overline{K}$ over $K$, the loop $\gamma'^{-1} \circ ({}^\sigma\gamma')$ is homotopy trivial. This implies that $T_\gamma^{\mathcal{F}}(f) \in \mathcal{F}_y^\nabla \otimes_K \mathbf{K}$ □

## 3.11 The comparison theorem of G. Herz

| | |
|---|---|
| $K$ | a finite extension of $\mathbb{Q}_p$ |
| $V$ | its ring of integers |
| $X$ | a Mumford curve over $K$ of genus $g$ (with semi-stable reduction ) |
| $X^{rig}$ | the rigidification of $X$ |
| $\mathcal{X}$ | the minimal regular model of $X$ |

We present G. Herz's ([Her05]) comparison between Faltings work "semi-stable vector bundles on Mumford curves" (Theorem 3.50) and DeWe parallel transport.

**Definition 3.65 (Herz).** Define by $\mathfrak{B}^{\mathcal{X}}_{X^{rig}}$ the full subcategory of all semi-stable rigid vector bundles of degree 0 on $X^{rig}_{\mathbb{C}_p}$ whose associated Faltings (Reversat- van der Put) representation (Definition 3.52) is isomorphic to a representation which has image in $GL_r(\mathfrak{o})$.

**Remark 3.66.** Herz defined a group $\pi_1^{ftop}(X^{an}, \overline{x})(\cong \hat{\mathbb{Z}}^g)$ that classifies finite analytic (topological) coverings of $X^{an}$ (see 3.6). This group is also the pro-finite completion of $\pi_1^{top}(X^{an}, \overline{x})(\cong \mathbb{Z}^g)$ (similar to the classical case). He further defined the pro-finite completion of a representation using the following fact

$$Hom(\mathbb{Z}^g, GL_r(\mathfrak{o})) = Hom_{cont}(\hat{\mathbb{Z}}^g, GL_r(\mathfrak{o})).$$

**Theorem 3.67 (Herz).** *Let $L \subset \mathbb{C}_p$ be a complete subfield of $\mathbb{C}_p$ which is an algebraic extension of $K$, and let $V'$ be its ring of integers. Let $E$ be a vector bundle in $\mathfrak{B}^{\mathcal{X}_{V'}}_{X_L^{rig}}$. Then the completed Faltings (Reversat- van der Put) representation attached to $E$ extended to $\mathbb{C}_p$ is isomorphic to the DeWe -representation attached to the algebraization of $E$.*

*Proof.* [Her05] Theorem 2.31 □

**Remark 3.68.** We will sketch his proof:

*Proof.* Let $\hat{\mathcal{E}}_\rho$ be the formal vector bundle on $\hat{\mathcal{X}}$ attached to an integral representation $\rho$ of $\pi_1^{top}(X^{an}_{\mathbb{C}_p}, \overline{x})$. Let $\mathcal{E}_\rho$ be the corresponding algebraic vector bundle on $\mathcal{X}$. For each $n \geq 1$ there exists an $G$ equivariant covering $\pi : \mathcal{Y} \to \mathcal{X}$ in $\mathcal{S}^{good}_{\mathcal{X}}$ for a group $G$ acting on $\mathcal{Y}$, such that the covering $\pi_K : \mathcal{Y}_K \to \mathcal{X}_K$ is finite étale and even analytic (topological) and Galois with group $G$.

$$\begin{array}{ccccccccc}
\pi_1^{alg}(X_{\mathbb{C}_p}, \overline{x}) & \xrightarrow{\phi_{y_i}} & Gal_{X_{\mathbb{C}_p}} Y_i & \xrightarrow{\phi_{\overline{y}}} & Gal_{X_{\mathbb{C}_p}} Y & \xrightarrow{=} & Aut^{op}_X Y & \xrightarrow{=} & G^{op} \\
| & & |^{\phi_{y_i'}} & \nearrow_{\phi_{\overline{y}'}} & & & & & \\
\pi_1^{ftop}(X^{an}_{\mathbb{C}_p}, \overline{x}) & \xrightarrow{\phi_{y_i'}} & Gal_{X_{\mathbb{C}_p}} Y_i'. & & & & & &
\end{array}$$

The reduction modulo $p^n$ of the DeWe representation attached to $\mathcal{E}_\rho$ denoted by $\rho_n^{DeWe}$ is the morphism

$$\rho_n^{DeWe} : G^{op} \longrightarrow Aut \mathcal{E}_{x_n}$$

$$\sigma \mapsto (y_n^*)^{-1} \sigma^* y_n^*$$

as in the following diagram where we abbreviated

$$H^0 := \Gamma(\mathcal{Y}_n, \pi_n^* \hat{\mathcal{E}}_n) = \Gamma(\mathcal{Y}_n, \pi_n^* \mathcal{E}_n)$$

$$\sigma \mapsto (\mathcal{E}_n \xrightarrow{(y_n^*)^{-1}} H^0 \xrightarrow{\sigma^*} H^0 \xrightarrow{y_n^*} \mathcal{E}_{x_n})$$

The morphism $\sigma^*$ is defined as follows:

$$H^0 \ni f \mapsto f \circ \sigma = \rho_n^{PuRe}(\sigma) f \in H^0$$

Then

$$\rho_n^{DeWe}(\sigma) = (y_n^*)^{-1} (\rho_n^{PuRe})(\sigma)(y_n^*)$$

Hence $\rho_n^{DeWe}$ and $\rho_n^{PuRe}$ are isomorphic representations. □

## 3.12 Galois Theory for schemes and commutative rings

We recall some facts about Galois theory for schemes and commutative rings. Most of the results we will need are in [Gro71] (especially Section V.2 pp.110-116 ), but are only stated for locally noetherian schemes. For schemes that are not necessarily locally noetherian we will refer to [Len97] (Galois Theory for schemes). For the Galois theory for commutative rings we refer to [ChHaRo65].

We recall the definition of Galois-coverings of (commutative) rings and schemes:

**Definition 3.69.**

a) Let $Y \to X$ be a finite étale covering of schemes, and let $G$ be a finite group of $X$-automorphisms of $Y$. The covering $Y/X$ is called Galois if the canonical morphism

$$Y \times_X G \to Y \times_X Y, \qquad (y, g) \mapsto (gy, y)$$

is an isomorphism

b) Let $A \to B$ be an étale covering of (commutative) rings, and let $G$ be a finite group of automorphisms of $B$ over $A$. We say that $B/A$ is Galois with group $G$ if the natural map

$$B \otimes_A B \to \prod_{g \in G} B, \qquad b \otimes c \mapsto (\ldots, b \cdot g(c), \ldots)$$

is an isomorphism.

**Remark 3.70.** There are many equivalent definitions of Galois étale covers of commutative rings or schemes. We refer the reader to [ChHaRo65] Theorem 1.3 page 18 for commutative rings, to [Gro71] Exp. V, Proposition 2.6, Definition 2.8 and Section 7 and to [Len97] Section 3.14. We just mention some characterizations of interest to us:

a) A covering $Y \to X$ (with a group $G$ of automorphisms) of schemes is Galois if and only if $Y$ is finite over $X$, $X = Y/G$, and the inertia groups at points of $Y$ are reduced to the identity ([Gro71] V Proposition 2.6 i))

b) If $X$ is connected, $Y \to X$ a finite étale covering, then the condition $X = Y/G$ is equivalent to the condition $Y(\Omega)/G = X(\Omega)$, for any algebraically closed extension $\Omega/K(x)$ for any point $x \in X$ ([Gro71] V Proposition 3.7 or [Len97] Section 3.14)

We will need the following descent lemma:

**Lemma 3.71.** *Let $B/A$ be an finite étale Galois covering with group $G$. Let $M$ be a $B$-module with semi-linear $G$-action. Then the natural map $s : B \otimes_A M^G \to M$ is an isomorphism.*

*Proof.* [ChHaRo65] Theorem 1.3 d) □

The following lemma is assumed to be well known but we did not find an exact reference in the literature. A similar statement can be found in [ChHaRo65] Lemma 1.7 page 21.

**Lemma 3.72.** *Let $B/A$ be an finite étale Galois covering with group $G$. Let $N$ be a $A$-module. Define a $G$-action on $B \otimes_A N$ by the rule $g(b \otimes n) := g(b) \otimes n$ for $g \in G$, $b \in B$, $n \in N$. Then the natural map*

$$B^G \otimes_A N \to (B \otimes_A N)^G \tag{2}$$

*is an isomorphism*

*Proof.* The natural map (2) is an isomorphism if and only if the natural morphism

$$B' \otimes_A B^G \otimes_A N \to B' \otimes_A (B \otimes_A N)^G \tag{3}$$

is an morphism because $B' := B$ is a faithfully flat $A$-algebra. We also have a canonical isomorphism $B' \otimes_A B^G \to (B' \otimes_A B)^G$ if we let $G$ act on $B'$ trivially. This can be seen using the following standard argument. The ring $B^G$ is the kernel of the exact sequence

$$0 \to B^G \to B \to \prod_{g \in G} B$$

where the last map sends $b$ to $(\ldots, g(b)-b, \ldots)$. This sequence remains exact if we tensor with the (faithfully) flat $A$-algebra $B'$:

$$0 \to B' \otimes_A B^G \to B' \otimes_A B \to \prod_{g \in G} (B' \otimes_A B).$$

The ring $(B' \otimes_A B)^G$ is by definition the kernel of this sequence, hence isomorphic to $B' \otimes_A B^G$. By the same reasoning we can rewrite the right hand side of (3) and obtain thus a morphism

$$(B' \otimes_A B)^G \otimes_A N \to (B' \otimes_A B \otimes_A N)^G. \tag{4}$$

Using simplification by $B'$ we can rewrite this as

$$(B' \otimes_A B)^G \otimes_{B'} (B' \otimes_A N) \to ((B' \otimes_A B) \otimes_{B'} (B' \otimes_A N))^G. \tag{5}$$

The extension $B/A$ is Galois, hence $(B' \otimes_A B)/B'$ is a trivial $G$-covering, by this we mean $B' \otimes_A B \cong \prod_{h \in H} B'$ where $H := G$ as a set and $G$ acts via permutations, i.e. $g((b_h)) := (b_{g(h)})$ if $(b_h) \in \prod_{h \in H} B'$. We can thus write

$$(\prod_{h \in H} B')^G \otimes_{B'} N' \to (\prod_{h \in H} N')^G. \tag{6}$$

where we abbreviated $N' := B' \otimes_A N$ and used that the tensor product respects direct sums on the right hand side. From this explicit description it follows that the morphism is an isomorphism □

**Lemma 3.73.** *Let $B/A$ be a finite étale Galois covering with group $G$. Let $A \to C$ be a commutative $A$-algebra and let $M$ be a $B$-module with semi-linear $G$-action. Let $G$ act on the tensor product $C \otimes_A M$ by the rule $g(c \otimes m) := c \otimes g(m)$. Then the canonical morphism*

$$C \otimes_A M^G \to (C \otimes_A M)^G$$

*is an isomorphism.*

*Proof.* By Lemma 3.71 we have an isomorphism of modules with semi-linear $G$-action $B \otimes_A M^G \cong M$. The we can rewrite the morphism above as

$$C \otimes_A M^G \to (B \otimes_A (C \otimes_A M^G))^G.$$

By definition the group $G$ acts trivially on the module $N := (C \otimes_A M^G)$. Hence we can apply Lemma 3.72 □

# 4 Flat vector bundles attached to representations

In this section we will explain how to attach a vector bundle with connection to a continuous $p$-adic representation of the étale fundamental group of a variety over a $p$-adic field in certain cases. This construction is classical and already carried out in a different context by H. Lange, U. Stuhler [LaSt77], R. Crew, N. Katz (Section 3.5), J. deJong [deJ95].

## 4.1 Vector bundles attached to representations of a finite group

| | |
|---|---|
| $R$ | a "base" ring (commutative with unit) |
| $\pi : Y \to X$ | a finite étale Galois cover of schemes with Group $G$ |
| $\mathbb{L}$ | a free $R$ module of rank $r$ |
| $\rho : G \to Aut_R(\mathbb{L})$ | a representation of $G$ on $\mathbb{L}$ |

In this section we recall in detail how to attach a vector bundle with connection to a representation of the Galois group of a finite étale cover of a scheme. The construction is classical, see for example [Mum70] Theorem 1 page 111 or [BoLuRa90] Section 6. By a vector bundle we mean a locally free sheaf of constant rank $r$.

**Remark 4.1.** If $W \subset Y$ is a $G$ invariant open subset, then we let $G$ act on $\mathcal{O}_Y(W)$ via the rule $g(f) := (g^{-1})^* f$ for all $f \in \mathcal{O}_Y(W)$.

**Construction 4.2.** -

a) The Group $G$ acts on the presheaf $\mathcal{F}' := \mathbb{L} \otimes_R \pi_* \mathcal{O}_Y$ via

$$g(v \otimes_R f) = \rho(g)(v) \otimes_R g(f) \qquad f \in \mathcal{O}_Y(\pi^{-1}(U)),\ v \in \mathbb{L},\ g \in G$$

for $U \subset X$ open. This $G$-action extends to the associated sheaf $\mathcal{F} := (\mathcal{F}')^\dagger$. Denote by $\mathcal{F}_\rho := \mathcal{F}^G$ the sheaf of elements fixed under the action of $G$.

b) Let $\rho_1$ and $\rho_2$ be two representation on two $R$-modules $\mathbb{L}_1$ and $\mathbb{L}_2$. Let $\varphi : \mathbb{L}_1 \to \mathbb{L}_2$ be a compatible map for the $G$-action. Then $\varphi$ induces a map

$$\mathbb{L}_1 \otimes_R \pi_* \mathcal{O}_Y \xrightarrow{\varphi \otimes id} \mathbb{L}_2 \otimes_R \pi_* \mathcal{O}_Y$$

which in turn induces a map $\mathcal{F}_{\rho_1} \xrightarrow{\varphi} \mathcal{F}_{\rho_2}$ because the map $\varphi \otimes id$ is $G$-equivariant.

**Proposition 4.3.** *The sheaf $\mathcal{F}_\rho$ defined in Construction 4.2 is a vector bundle of rank $r$ and $\pi^*\mathcal{F}_\rho \cong \mathcal{F}$*

*Proof.* The assertion is local so we may take $\mathrm{Spec}A \subset X$ and $\mathrm{Spec}B = \pi^{-1}(\mathrm{Spec}A)$ and we set $M := \mathbb{L} \otimes_R B$. The canonical map $B \otimes_A M^G \to M$ is an isomorphism by Lemma 3.71. The morphism $A \to B$ is faithfully flat and $B \otimes_A M^G \cong M$ is a free (hence projective) finitely generated $B$-module. We can then apply [Len97] Proposition 4.12 p. 64 to deduce that $M^G$ is a finitely generated projective $A$-module. The rank of $M^G$ is equal to $r$ because $M \cong B \otimes_A M^G$ has rank $r$. □

We will now construct a connection

$$\nabla_\rho : \mathcal{F}_\rho \to \mathcal{F}_\rho \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathrm{Spec}R}$$

attached to $\rho$ and $\mathcal{F}_\rho$:

For $U \subset X$ open let us define a morphism (the "constant" connection)

$$\nabla : (\mathbb{L} \otimes_R \pi_*\mathcal{O}_Y)(U) \to ((\mathbb{L} \otimes_R \pi_*\mathcal{O}_Y) \otimes_{\pi_*\mathcal{O}_Y} \pi_*\Omega^1_{Y/R})(U)$$

$$\nabla(v \otimes_R f) := v \otimes_R df \qquad \text{for } v \in \mathbb{L}, f \in \mathcal{O}_Y(\pi^{-1}U).$$

(here the tensor product is the tensor product of presheaves) These morphisms define a morphism of presheaves and of the corresponding associated sheaf. For any $g \in G$ we have

$$\nabla(g(v \otimes_R f)) = \nabla(\rho(g)(v) \otimes_R g(f)) = \rho(g)(v) \otimes_R dg(f) =$$

$$= \rho(\sigma)(v) \otimes_R g(df) = g(\nabla(v \otimes_R f)).$$

In other words the map $\nabla$ is $G$-equivariant. We claim that it descends to a map $\mathcal{F}_\rho \to \mathcal{F}_\rho \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathrm{Spec}R}$ of vector bundles on $X$. The assertion is local so we may take $\mathrm{Spec}A \subset X$ and $\mathrm{Spec}B = \pi^{-1}(\mathrm{Spec}A)$ and we set $M := \mathbb{L} \otimes_R B$. Because $A \to B$ is étale there is a canonical isomorphism $B \otimes_A \Omega^1_{A/R} \cong \Omega^1_{B/R}$ ([EGA] IV (Quatriéme partie) No. 32, Corollaire 17.2.4.). We have to show that the canonical morphism

$$M^G \otimes \Omega^1_{A/R} \to ((M^G \otimes_A B) \otimes_B (B \otimes_A \Omega^1_{A/R}))^G \tag{7}$$

is an isomorphism. We can simplify the right hand side by $B$ and change the ordering of $M^G$ and $B$:

$$(B \otimes_A (M^G \otimes_A \Omega^1_{A/R}))^G$$

Now this is a tensor product of $B$ with a module with trivial $G$-action. By Lemma 3.72 the module of $G$ invariants is $A \otimes_A (M^G \otimes_A \Omega^1_{A/R})$ which is the left hand side of (7) what we wanted to show.

## 4.2 Vector bundles attached to continuous representations of algebraic fundamental groups

| | |
|---|---|
| $\lambda : \mathcal{X} \to \mathrm{Spec}\overline{\mathbb{Z}_p}$ | a projective, integral, normal and flat $\overline{\mathbb{Z}_p}$-scheme |
| $x$ | a $\overline{\mathbb{Q}_p}$-valued base point of $\mathcal{X}$ |
| $\rho : \pi_1^{alg}(\mathcal{X}, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ | a continuous representation on a free $\mathfrak{o}$-module $\mathbb{L}$ of rank $r$ |

In this section we attach a vector bundle with connection to a continuous representation of the algebraic fundamental group of $\mathcal{X}$. We use the previous construction modulo $p^n$, then we apply the inductive method of liftings. This is a variant of Katz/Crews construction (Section 3.5).

**Remark 4.4.** a) An example of a scheme $\mathcal{X}$ that satisfies the properties above is a smooth integral projective scheme over $\overline{\mathbb{Z}_p}$. Another example arises when one considers $\mathcal{X} = \mathcal{X}' \times_{V'} \overline{\mathbb{Z}_p}$ where $\mathcal{X}'$ is a semi-stable curve over the ring of integers $V'$ of a finite extension $K'$ of $\mathbb{Q}_p$, such that the generic fiber $X'$ of $\mathcal{X}'$ is smooth and geometrically integral. The generic fiber $X$ of $\mathcal{X}$ is integral and $\overline{\mathbb{Z}_p}$ is integral, hence $\mathcal{X}$ (flat over $\overline{\mathbb{Z}_p}$) is also integral by [Liu02] Proposition 4.3.8. If $X'$ is smooth over $K'$ then the same is true for every finite extension of $K'$, and hence $X' \times_{K'} K''$ is normal for every finite extension $K''$ of $K'$. By [Liu02] Proposition 10.3.15 every scheme $\mathcal{X}' \times_{V'} V''$ is normal when $V''$ is the ring of integers of a finite extension $K''$ of $K'$. This implies that $\mathcal{X}$ is also normal, because equations for integral elements can always be defined over a finite extension of $\mathbb{Q}_p$. For a more general discussion of such models we refer the reader to [Fal02] Remark 5 and page 205

b) Let $\mathcal{Y} \to \mathcal{X}$ be a finite étale covering with $\mathcal{Y}$ connected. Then $\mathcal{Y}$ is also integral and normal by [Gro71] I Corollaire 9.10, 9.11 and Proposition 10.1. This will allow us to apply the Galois theory of (integral) commutative rings in [ChHaRo65] or [Len97] Corollary 6.17.

c) Because $\mathcal{X}_{\mathfrak{o}}$ is projective, the structure sheaf of $\widehat{\mathcal{X}}_{\mathfrak{o}}$ is coherent. This is true because locally $\mathcal{X}_{\mathfrak{o}}$ can be covered by $\mathfrak{o}$-algebras $A_i$ of finite type whose completion is of topologically finite representations (see Section 3.7).

**Construction 4.5.** We will use freely the formal GAGA results from Section 3.7.

Consider the $\rho : \pi_1^{alg}(\mathcal{X}, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$. The representation modulo $p^n$

$$\rho_n : \pi_1^{alg}(\mathcal{X}, x) \to Aut_{\mathfrak{o}}(\mathbb{L}) \to Aut_{\mathfrak{o}_n}(\mathbb{L}_n)$$

factors over a finite group $G$, because $\pi_1^{alg}(\mathcal{X},x)$ is pro-finite and $Aut_{\mathfrak{o}_n}(\mathbb{L}_n)$ is discrete. By Remark 3.24 there exists a finite étale Galois covering $\pi : \mathcal{Y} \to \mathcal{X}$ such that $G$ can be identified (up to conjugation) with $Aut_{\mathcal{X}}\mathcal{Y}$. We can use the construction of Section 4.1 to construct a vector bundle with connection $(\mathcal{F}_{\rho_n}, \nabla_{\rho_n})$ on $\mathcal{X}_{\mathfrak{o},n}$. The pairs $(\mathcal{F}_{\rho_n}, \nabla_{\rho_n})$ define (for varying $n$) a projective system of vector bundles with connection (see Lemma 4.6 below for the projectivity). The projective limit

$$(\widehat{\mathcal{F}_\rho}, \widehat{\nabla_\rho}) := \lim_{\leftarrow}(\mathcal{F}_{\rho_n}, \nabla_{\rho_n})$$

is a formal vector bundle with connection on the formal scheme $\hat{\mathcal{X}_{\mathfrak{o}}}$ (the $p$-adic completion of $\mathcal{X}_{\mathfrak{o}}$). Because $\mathcal{X}_{\mathfrak{o}}$ is projective we can apply the formal GAGA - Theorem and obtain an algebraic vector bundle with connection $(\mathcal{F}_\rho, \nabla_\rho)$ on $\mathcal{X}_{\mathfrak{o}}$. If $\varphi$ is a morphism between two representations of $\pi_1^{alg}(\mathcal{X},x)$, then one applies the construction in Section 4.1 inductively modulo $p^n$ to obtain a morphism between the corresponding vector bundles with connection.

We now check that the system of vector bundles and connection is in fact projective:

**Lemma 4.6.** *The system of pairs $(\mathcal{F}_{\rho_n}, \nabla_{\rho_n})$ forms a projective system, i.e.*

$$(\mathcal{F}_{\rho_n}, \nabla_{\rho_n}) \cong (v_{n,n+1})^*(\mathcal{F}_{\rho_{n+1}}, \nabla_{\rho_{n+1}})$$

*where $v_{n,n+1} : \mathcal{X}_n \to \mathcal{X}_{n+1}$ is the reduction map.*

*Proof.* Let $G_{n+1}, G_n$ be the corresponding groups and denote by $H$ the kernel of $G_{n+1} \twoheadrightarrow G_n$. The assertion is local, so let $\mathrm{Spec}A \subset \mathcal{X}_{\mathfrak{o}}$ be an affine open and let $B/A$ resp. $C/A$ be the corresponding finite étale Galois-coverings with group $G_n$ resp. $G_{n+1}$. Let us also set $M_n := \mathcal{F}_{\rho_n}(\mathrm{Spec}A_n)$ and $M_{n+1} := \mathcal{F}_{\rho_{n+1}}(\mathrm{Spec}A_{n+1})$. Because $A, B, C$ are integral (Remark 4.4) we can apply the Galois theory of commutative rings [ChHaRo65] Section 2.2 pages 22-24 or [Len97] Corollary 6.17 to deduce that $C/B$ is Galois with group $H$. If we set $A_n := A \otimes_{\mathfrak{o}} \mathfrak{o}_n$, $B_n := B \otimes_A A_n$, $C_n := C \otimes_A A_n$ then $C_n/A_n$, $C_n/B_n$ and $B_n/A_n$ are also Galois with groups $G_{n+1}$, $H$ and $G_n$ ([ChHaRo65] Lemma 1.7 page 21). We want to compute

$$A_n \otimes_{A_{n+1}} M_{n+1} = A_n \otimes_{A_{n+1}} (C_{n+1} \otimes_{\mathfrak{o}} \mathbb{L}_{n+1})^{G_{n+1}}.$$

Because of Lemma 3.73 we can rewrite this as

$$(A_n \otimes_{A_{n+1}} C_{n+1} \otimes_{\mathfrak{o}} \mathbb{L}_{n+1})^{G_{n+1}}.$$

Because $H$ is a normal subgroup of $G_{n+1}$ with quotient $G_n$ we can rewrite this as

$$((A_n \otimes_{A_{n+1}} C_{n+1} \otimes_{\mathfrak{o}} \mathbb{L}_{n+1})^H)^{G_n}.$$

For the $H$-invariants we can also write (replacing $n+1$ by $n$ everywhere and using simplification by $A_n$ and $B_n$)

$$(C_n \otimes_{B_n} (B_n \otimes_{\mathfrak{o}} \mathbb{L}_n))^H.$$

Now $H$ acts on $C_n$ and trivially on $B_n \otimes_{\mathfrak{o}} \mathbb{L}_n$ (note that $\rho_n$ is trivial on $H$), thus by Lemma 3.72 the module of $H$-invariants is isomorphic to $B_n \otimes_{\mathfrak{o}} \mathbb{L}_n$ (using simplification by $B_n$ again). The group $G_n$ acts on this module by its usual action on $B_n$ and via $\rho_n$ on $\mathbb{L}$. By definition the module of $G_n$-invariants of this module is isomorphic to $M_n$. The connections $\nabla_n$ resp. $\nabla_{n+1}$ are by definition induced from the "constant" connection on $B \otimes_{\mathfrak{o}} \mathbb{L}_n$ resp. $C \otimes_{\mathfrak{o}} \mathbb{L}_{n+1}$ and the "constant" connection on the first module is compatible with the "constant" connection on the second modulo $p^n$. □

**Definition 4.7.** We denote the category of vector bundles attached to continuous representations $\rho : \pi_1^{alg}(\mathcal{X}, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ on free finitely generated $\mathfrak{o}$-modules $\mathbb{L}$ by $\mathfrak{B}^{rep}_{\mathcal{X}_{\mathfrak{o}}}$.

Let $(\mathcal{F}, \nabla)$ be a vector bundle with connection on $\mathcal{X}_{\mathfrak{o}}$. Assume that for any $n \geq 1$ the sheaf $\mathcal{F}_n$ (the reduction modulo $p^n$ of $\mathcal{F}$) can be étale trivialized. I.e. there exists a finite étale covering $\mathcal{Y} \to \mathcal{X}$ which satisfies $\pi_n^*(\mathcal{F}_n, \nabla_n) \cong (\mathcal{O}_{\mathcal{Y}_n}, d_{\mathcal{Y}_n})^r$.

**Assumption 4.8.** For all $n \geq 1$ and for all finite étale Galois covers $\mathcal{Y} \to \mathcal{X}$ the following holds:

$$H^0(\mathcal{Y}_{\mathfrak{o},n}, \mathcal{O}_{\mathcal{Y}_{\mathfrak{o},n}}) \cong \mathfrak{o}_n.$$

To each vector bundle $\mathcal{F}$ satisfying this properties one can use the construction of Deninger and Werner (Sections 3.2 and 3.3) and attach a representation $\rho_{\mathcal{F}}$ of $\pi_1^{alg}(\mathcal{X}, x)$ to the vector bundle $\mathcal{F}$.

**Remark 4.9.** The Assumption (4.8) on $\mathcal{X}$ is satisfied if $\mathcal{X}$ is induced by base change with $\overline{\mathbb{Z}_p}$ from a semi-stable curve that is defined over the ring of integers of a finite extension of $\mathbb{Q}_p$. To see this note that finite étale covers of semi-stable curves are semi-stable [Liu02] Exercise 3.9 page 529 and the assumption is true for semi-stable curves (after base change with $\overline{\mathbb{Z}_p}$) as shown in [DeWe05b] Theorem 1(1).

Our construction is compatible with Faltings' construction in Section 3.4, especially Lemma 3.13:

**Remark 4.10.** Let $V$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let $\mathcal{X}'$ be a proper $V$-scheme with toroidal singularities (e.g. $\mathcal{X}'$ is smooth or semi-stable). Assume that the scheme $\mathcal{X}$ satisfies $\mathcal{X} = \mathcal{X}' \otimes_V \overline{\mathbb{Z}_p}$. Let $\mathrm{Spec}R \subset \mathcal{X}'$ be a small affine open. The inclusion $U := \mathrm{Spec}R_1 \subset \mathcal{X}$ induces a morphism of fundamental groups $\pi_1^{alg}(U) \to \pi_1^{alg}(\mathcal{X})$, and $\pi_1^{alg}(U)$ is isomorphic to the Galois group of the maximal étale extension $R_1^{et}$ of $R_1$ (i.e. the union of all finite étale extensions of $R_1$) by [Len97] 6 Corollary 17. We define $\Delta^{et} := Gal(R_1^{et}/R_1)$. Consider the following sequence

$$\Delta = Gal(\overline{R}/R_1) \to Gal(R_1^{et}/R_1) \cong \pi_1^{alg}(U) \to \pi_1^{alg}(\mathcal{X}) \xrightarrow{\rho} Aut_{\mathfrak{o}}(\mathbb{L}) \qquad (8)$$

(here the first map is the restriction of $R_1$-automorphisms of $\overline{R}$ to $R_1^{et}$). We need to assume, that the representation $\rho$ is trivial modulo $p^{2\alpha}$ as in Lemma 3.13. The module

$$\overline{M} := \overline{R} \otimes_{\overline{\mathbb{Z}_p}} \mathbb{L}_n$$

can be equipped with a semi-linear $\Delta$ operation, if we let $\Delta$ act via the diagonal action, i.e. $\Delta$ acts on $\overline{R}$ via the obvious action and on $\mathbb{L}_n$ via the diagram (8). The generalized representation $\overline{M}$ is induced from the $R_1^{et}$-module

$$M_1^{et} := R_1^{et} \otimes_{\overline{\mathbb{Z}_p}} \mathbb{L}_n$$

equipped with the $\Delta_{et}$ diagonal action by diagram (8). Let $S \subset R_1^{et}$ be the finite étale Galois extension (with group $G$) of $R_1$ that trivializes $\rho_n$. The module

$$M_1 := (S \otimes_{\overline{\mathbb{Z}_p}} \mathbb{L}_n)^G$$

is by definition isomorphic to $\mathcal{F}_{\rho,n}(\mathrm{Spec}R_1)$, the module of Construction 4.5 modulo $p^n$. The module $M_1$ is projective over $R_1/p^n$ of rank $r$, and induces an isomorphism of $R_1^{et} - \Delta^{et}$-modules $M_1^{et} \cong R_1^{et} \otimes_{R_1} M_1$ . If we equip $M_1$ with the trivial $\Delta_\infty$ action, then there is an isomorphism of modules equipped with a semi-linear $\Delta$-action

$$\overline{M} = \overline{R} \otimes_{\overline{\mathbb{Z}_p}} \mathbb{L}_n \cong \overline{R} \otimes_{R_1} M_1. \qquad (9)$$

(The $\Delta$-action on the right hand side is given by the obvious action on $\overline{R}$ and by the (trivial) $\Delta$-action on $M_1$ induced by $\Delta \to \Delta_\infty$). If $M_1'$ is the $\Delta_\infty$-module of Lemma 3.13 a) associated to the generalized representation $\overline{M}$, then $M_1'$ and $M_1$ are almost isomorphic (as $R_1 - \Delta_\infty$ modules) modulo $p^{n-\alpha}$ if $\overline{R}/R_1$ is almost faithfully flat. This is true (assuming that $\overline{R}/R_1$ is almost faithfully flat), because their associated generalized representations are isomorphic (9) and one can apply Lemma 3.13 b) to get a morphism (modulo $p^{n-\alpha}$) between the two, which is actually an almost isomorphism (modulo $p^{n-\alpha}$) by almost faithfully flat descent for $\overline{R}/R_1$ (we assume that $\overline{R}/R_1$ is almost faithfully flat, which would follow from the explicit description of $R_\infty/R_1$ and Faltings' almost purity theorem ([Fal02] Theorem 4) for $\overline{R}/R_\infty$). Because the action of $\Delta_\infty$ on $M_1$ is trivial, the associated Higgs field $\theta$ is zero.

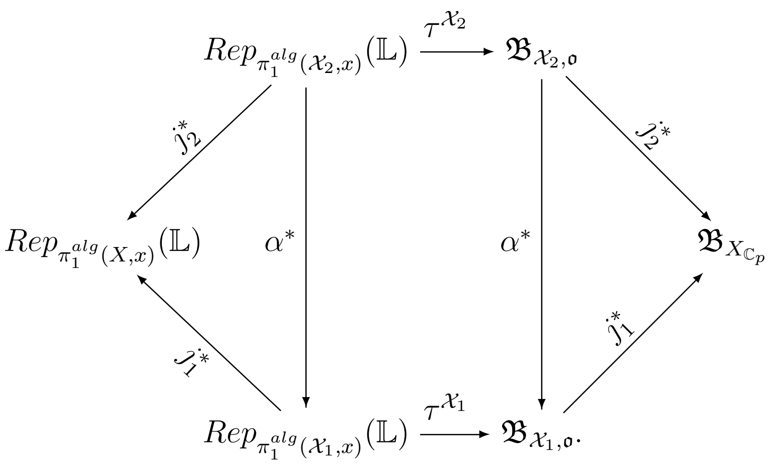
## 4.3 Compatibility for different models

| | |
|---|---|
| $X/\mathrm{Spec}\overline{\mathbb{Q}_p}$ | a smooth projective variety |
| $\mathrm{Spec}\overline{\mathbb{Q}_p} = x \to X$ | a base point of $X$ |

Let $\mathcal{X}$ be a projective, integral, normal, flat $\overline{\mathbb{Z}_p}$-model of $X$ and let $j : X \to \mathcal{X}$ be the canonical open immersion. Let $\rho : \pi_1^{alg}(X, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ be a representation on a free $\mathfrak{o}$-module $\mathbb{L}$ of rank $r$, such that $\rho$ factors over $\pi_1^{alg}(\mathcal{X}, x)$. Then we obtain a vector bundle with connection $(F_\rho, \nabla_\rho) := j^*(\mathcal{F}_\rho, \nabla_\rho)$ on $X_{\mathbb{C}_p}$ using the construction of Section 4.2. We will show that this construction is compatible for different models. In the case that $X$ is a curve, two models are dominated by a third one, and therefore this construction does not depend on a chosen model.

**Proposition 4.11.** *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two projective, integral, connected and flat $\overline{\mathbb{Z}_p}$ - models of $X$, and let $\alpha : \mathcal{X}_1 \to \mathcal{X}_2$ be a morphism restricting to the identity on the generic fiber.*

*Then the following diagram is commutative:*




*In this diagram the two $\alpha^*$ are the obvious pullback maps, and $\tau$ is the map attaching a bundle with connection to a representation studied in Section 4.2.*

We divide the proof in three lemmas:

**Lemma 4.12.** *Let $\rho \in Rep_{\pi_1^{alg}(\mathcal{X}_{2},x)}(\mathbb{L})$ be a representation. Then there is a canonical isomorphism*

$$\alpha^*(\mathcal{F}_\rho, \nabla_\rho) \cong (\mathcal{F}_{\alpha^*\rho}, \nabla_{\alpha^*\rho})$$

*(In this lemma the assumption that $\alpha$ restricts to the identity on the generic fiber is not needed).*

*Proof.* It suffices to proof the assertion modulo $p^n$ i.e.:

$$\alpha_n^*(\mathcal{F}_{\rho_n}, \nabla_{\rho_n}) \cong (\mathcal{F}_{\alpha^*\rho_n}, \nabla_{\alpha^*\rho_n})$$

Let $cov_2 : \mathcal{Y}_n \to \mathcal{X}_{2,n}$ be a finite étale Galois cover trivializing $\rho_n$ with Galois-group $G = Aut(\mathcal{Y}_n/\mathcal{X}_{2,n})$. Consider the fiber product

$$\begin{array}{ccc} \mathcal{W}_n := \mathcal{X}_{1,n} \times_{\mathcal{X}_{2,n}} \mathcal{Y}_n & \xrightarrow{pr_2} & \mathcal{Y}_n \\ cov_1 \downarrow G & & cov_2 \downarrow G \\ \mathcal{X}_{1,n} & \xrightarrow{\alpha} & \mathcal{X}_{2,n} \end{array}$$

(The covering $cov_1$ is also an étale Galois cover with automorphism group $G$ (see [ChHaRo65] Lemma 1.7 page 21 for the affine case))

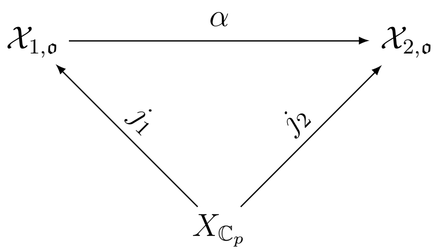
The assertion is local, so let $\mathrm{Spec}A_2 \subset \mathcal{X}_{2,n}$, $\mathrm{Spec}A_1 \subset \mathcal{X}_{1,n}$ affine open subsets satisfying $\alpha(\mathrm{Spec}A_1) \subset \mathrm{Spec}(A_2)$, and let $\mathrm{Spec}B := cov_2^{-1}(\mathrm{Spec}A_2)$. Then we have to show that the canonical map

$$A_1 \otimes_{A_2} (B \otimes_{\mathfrak{o}_n} \mathbb{L}_n)^G \to ((A_1 \otimes_{A_2} B) \otimes_{\mathfrak{o}_n} \mathbb{L}_n)^G$$

is an isomorphism. If we set $M := B \otimes_{\mathfrak{o}_n} \mathbb{L}_n$, $A := A_2$ and $C := A_1$ then we can apply Lemma 3.73. This shows the assertion for the vector bundles. The "constant" connection on $\mathcal{O}_{\mathcal{W}_{\mathfrak{o},n}} \otimes_{\mathfrak{o}} \mathbb{L}_n$ is equal to the pullback of the "constant" connection on $\mathcal{O}_{\mathcal{Y}_{\mathfrak{o},n}} \otimes_{\mathfrak{o}} \mathbb{L}_n$. This implies that the connections restricted to the $G$-invariant elements are compatible □

**Lemma 4.13.** *Let $(\mathcal{F}_1, \nabla_1)$ and $(\mathcal{F}_2, \nabla_2)$ be two vector bundles with connections on $\mathcal{X}_{1,\mathfrak{o}}$ and $\mathcal{X}_{2,\mathfrak{o}}$ that satisfy $\alpha^*(\mathcal{F}_2, \nabla_2) = (\mathcal{F}_1, \nabla_1)$. Then $(F_1, \nabla_1) := j_1^*(\mathcal{F}_1, \nabla_1)$ and $(F_2, \nabla_2) := j_2^*(\mathcal{F}_2, \nabla_2)$ are isomorphic vector bundles with connection.*

*Proof.* The following diagram is commutative



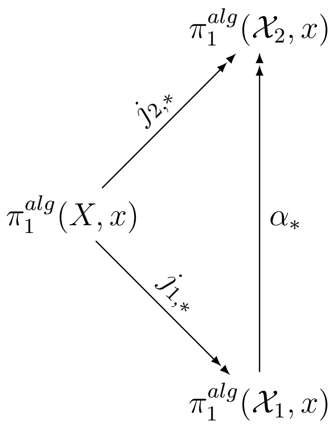

hence

$$F_1 = j_1^*\mathcal{F}_1 \cong j_1^*\alpha^*\mathcal{F}_2 \cong (\alpha \circ j_1)^*\mathcal{F}_2 = j_2^*\mathcal{F}_2 = F_2$$

and by the same reasoning $\nabla_1 \cong \nabla_2$ □

**Lemma 4.14.** *Let $\rho_1$ and $\rho_2$ be two representations of $\pi_1^{alg}(\mathcal{X}_1, x)$ and $\pi_1^{alg}(\mathcal{X}_2, x)$ respectively. Assume that pullback induces identical representation of $\pi_1^{alg}(X, x)$, i.e. $j_1^*\rho_1 = j_2^*\rho_2$. Then $\alpha^*\rho_2 \cong \rho_1$.*

*Proof.* We have $\alpha \circ j_1 = j_2$. This induces a commutative diagram between the fundamental groups:




The two maps on the left hand side are surjective, because the functor $H$ that maps finite étale covers of $\mathcal{X}_i$ $(i = 1, 2)$ to the generic fiber maps connected covers to connected covers and one can apply [Gro71] V Proposition 6.9. The pullback maps for representations satisfy $j_1^* \circ \alpha^* \cong j_2^*$. By assumption $j_1^*\rho_1 = j_2^*\rho_2 \cong j_1^* \circ \alpha^*\rho_2$. The morphism $j_{1,*} : \pi_1^{alg}(X, x) \to \pi_1^{alg}(\mathcal{X}_1, x)$ is surjective, hence the corresponding map $j_1^*$ on representations is injective. This implies $\rho_1 \cong \alpha^*\rho_2$ □

The following lemma from [DeWe05b] shows that in the case of curves, that two arbitrary models are dominated by a third having nice properties.

**Lemma 4.15.** *Let $X$ be a smooth projective curve over $\overline{\mathbb{Q}_p}$ and let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two projective, integral, normal and flat $\overline{\mathbb{Z}_p}$-models of $X$. Then there is a third (projective, integral, normal, flat) model $\mathcal{X}_3$ of $X$ together with morphisms*

$$\mathcal{X}_1 \xleftarrow{p_1} \mathcal{X}_3 \xrightarrow{p_2} \mathcal{X}_2$$

*restricting to the identity on the generic fibers (after their identification with $X$).*

*Proof.* That two models can be (strictly) dominated by a third one is proven [DeWe05b] Proposition 27, page 586. The model can be chosen to be semi-stable by [DeWe05b] Theorem 1, page 556. If a semi-stable model of $X$ is already defined over the ring of integers of a finite extension of $\mathbb{Q}_p$, then we can apply Lipmans' resolution of singularities to obtain a regular semi-stable model, which is projective by a theorem of Lichtenbaum. But a semi-stable $\overline{\mathbb{Z}_p}$ model of $X$ is defined over a the ring of integers $V$ of a finite extension of $\mathbb{Q}_p$, and the base change with $\mathrm{Spec}\overline{\mathbb{Z}_p}$ of a projective $V$-model is a projective $\overline{\mathbb{Z}_p}$-model. This model is also integral, normal and flat as was shown in Remark 4.4 □

**Theorem 4.16.** *Let $X$ be a smooth projective curve over $\overline{\mathbb{Q}_p}$ and let $\mathcal{X}_1$ and $\mathcal{X}_2$ be projective, integral, normal and flat $\overline{\mathbb{Z}_p}$-models of $X$ and let $j_1$, $j_2$ be the corresponding open immersions. Let*

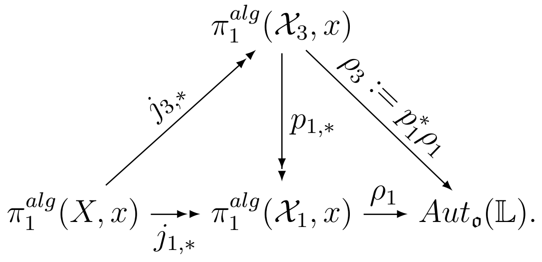
$$\rho : \pi_1^{alg}(X, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$$

*be a continuous representation on a free $\mathfrak{o}$-module $\mathbb{L}$ of rank $r$. Assume that $\rho$ factors over $\pi_1^{alg}(\mathcal{X}_1, x)$ and $\pi_1^{alg}(\mathcal{X}_2, x)$. Denote the corresponding representations of these groups $\rho_1$ and $\rho_2$. Set $(F_1, \nabla_1) := j_1^*(\mathcal{F}_{\rho_1}, \nabla_{\rho_1})$ and $(F_2, \nabla_2) := j_2^*(\mathcal{F}_{\rho_2}, \nabla_{\rho_2})$. Then both vector bundles with connection are isomorphic*

$$(F_1, \nabla_1) \cong (F_2, \nabla_2)$$

*In other words the association $\rho \mapsto j^*(\mathcal{F}_\rho, \nabla_\rho)$ does not depend on the choice of a projective model $\mathcal{X}$.*

*Proof.* Choose a projective model $\mathcal{X}_3$ of $X$ as in Lemma 4.15 and denote the canonical open immersion by $j_3$. Then there exists a commutative diagram



By Proposition 4.11 the vector bundles $j_1^*\mathcal{F}_{\rho_1}$ and $j_3^*\mathcal{F}_{\rho_3}$ are isomorphic on $X_{\mathbb{C}_p}$. The same is true for $j_2^*\mathcal{F}_{\rho_2}$ if one replaces 1 by 2. The same reasoning applies to the connections □

## 4.4 Vector bundles attached to temperate representations

| | |
|---|---|
| $X$ | a smooth projective $\overline{\mathbb{Q}_p}$-variety |
| $\pi : Y \to X$ | a finite connected étale Galois covering (with group G) |
| $\mathcal{Y}$ | a projective, integral, normal, flat $\overline{\mathbb{Z}_p}$-model of $Y$ |

From Section 4.2 we already know how to attach a vector bundle with connection to a representation of the étale fundamental group of $\mathcal{Y}$. Using descent theory we can attach a vector bundle to a representation of the fundamental group of $X$ that factors over the fundamental group of $\mathcal{Y}$ when restricted to the fundamental group of $Y$. We call a representation that satisfies this property temperate. In [And03] Chapter III (see also 3.6) André calls a Berkovich étale covering temperate if it decomposes as a finite étale covering and a topological covering. (The relation between these two terminologies will be explained in Section 5)

**Definition 4.17 (temperate representations).** - Let $\rho : \pi_1^{alg}(X, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ be a continuous representation of the étale fundamental group of $X$ with base point $x$ on a finitely generated free $\mathfrak{o}$-module $\mathbb{L}$. The representation $\rho$ is called temperate if there is a commutative diagram

$$\begin{array}{ccc} \pi_1^{alg}(Y, y) & \overset{can}{\twoheadrightarrow} & \pi_1^{alg}(\mathcal{Y}, y) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow \\ \pi_1^{alg}(X, x) & \xrightarrow{\rho} & Aut_{\mathfrak{o}}(\mathbb{L}) \end{array}$$

where $\pi : Y \to X$ is a finite étale Galois covering with group $G$, $y = \mathrm{Spec}\overline{\mathbb{Q}_p}$ a point above $x$ and $\mathcal{Y}$ a projective, integral, normal, flat $\overline{\mathbb{Z}_p}$ model of $Y$.

**Remark 4.18.** a) In this definition the $\overline{\mathbb{Z}_p}$-model $\mathcal{Y}$ is unspecified. We say that $\rho$ is temperate with respect to a certain kind of model if there is a diagram as above with $\mathcal{Y}$ a certain kind of model (e.g minimal regular model, Néron model). In the case of abelian varieties with good reduction we will restrict us for simplicity to $N$-covers and Néron models. If $Y$ is a smooth projective curve then we require that the $G$-action on $Y$ extends to $\mathcal{Y}$. This is the case for minimal regular models that are defined over the valuation ring of a finite extension of $\mathbb{Q}_p$ [Liu02] (Proposition 9.3.13). Also every model of $Y$ can be dominated by a model such that the $G$-action extends ([Liu06] proof of Lemma 2.4).

b) One could define temperate representations more generally for arbitrary proper models $\mathcal{Y}$. If $\mathcal{Y}'$ is another model of $Y$ that strictly dominates $\mathcal{Y}$ (being the identity on $Y$) then $\rho$ it is also temperate with respect to $\mathcal{Y}'$, this can be seen from the proof of Lemma 4.14.

The following proposition will be useful to compare two temperate representations:

**Proposition 4.19.** *Let $X/Spec\overline{\mathbb{Q}_p}$ be a smooth proper curve or an abelian variety with good reduction. Let $x$ be a base point of $X$ and let $\rho_i : \pi_1^{alg}(X,x) \to Aut_{\mathfrak{o}}(\mathbb{L}_i)$ ($i = 1,2$) be two temperate representations on finite dimensional $\mathfrak{o}$-modules. Then $\rho_1, \rho_2$ are temperate with respect to a common covering and model. To be precise there exists a finite étale Galois covering $\pi : Y \to X$ with group $G$, a point $y$ above $x$ and $\mathcal{Y}$ a projective, integral, normal and flat model of $Y$ such that there exists commutative diagram*

$$\begin{array}{ccc} \pi_1^{alg}(Y,y) & \xrightarrow{can} & \pi_1^{alg}(\mathcal{Y},y) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \tilde{\rho}_i} \\ \pi_1^{alg}(X,x) & \xrightarrow{\rho_i} & Aut_{\mathfrak{o}}(\mathbb{L}) \end{array}$$

*for $i = 1,2$ and representations $\tilde{\rho}_i$.*

*Proof.* We first consider the case when $A := X$ is an abelian variety with smooth Néron model $\mathcal{A}$. The two representations $\rho_1, \rho_2$ are temperate, hence there exists positive integers $N_1, N_2$ such that there exists commutative diagrams as in Definition 4.17 with $Y_i = A$, $\mathcal{Y}_i = \mathcal{A}$ and $\pi = N_i$ for $i = 1,2$. We claim that we can choose a common $N = N_1 = N_2$ satisfying the same properties. Set $N := N_1 \cdot N_2$ and consider the following commutative diagram of abelian schemes and their corresponding fundamental groups:

$$\begin{array}{ccccccc}
\mathcal{A} & \xleftarrow{can} & A & \qquad & \pi_1^{alg}(\mathcal{A},0) & \xleftarrow{can} & \pi_1^{alg}(A,0) \\
\downarrow N_1 & & \downarrow N_1 & & \downarrow N_{1,*} & & \downarrow N_{1,*} \\
\mathcal{A} & \xleftarrow{can} & A & & \pi_1^{alg}(\mathcal{A},0) & \xleftarrow{can} & \pi_1^{alg}(A,0) \\
\downarrow N_2 & & \downarrow N_2 & & \downarrow N_{2,*} & & \downarrow N_{2,*} \\
\mathcal{A} & \xleftarrow{can} & A & & \pi_1^{alg}(\mathcal{A},0) & \xleftarrow{can} & \pi_1^{alg}(A,0)
\end{array}$$

The representation $\rho_2$ (restricted to $N_{2,*}\pi_1^{alg}(A,0)$) factors as

$$\pi_1^{alg}(A,0) \stackrel{can}{\to} \pi_1^{alg}(\mathcal{A},0) \stackrel{\tilde{\rho}_2}{\to} Aut_{\mathfrak{o}}(\mathbb{L}).$$

From the commutative diagram it follows that $\rho_2$ (restricted to $N_{2,*} \circ N_{1,*}\pi_1^{alg}(A,0)$) factors also as

$$\pi_1^{alg}(A,0) \stackrel{can}{\to} \pi_1^{alg}(\mathcal{A},0) \stackrel{N_{1,*}}{\to} \pi_1^{alg}(\mathcal{A},0) \stackrel{\tilde{\rho}_2}{\to} Aut_{\mathfrak{o}}(\mathbb{L})$$

This implies that $\rho_2$ is also a temperate representation with the $N$ covering instead of $N_2$. The same reasoning applies to $\rho_1$ because we can interchange 1 and 2. If $X$ is a curve with finite étale Galois coverings $Y_1$ and $Y_2$, then we can take a finite étale Galois covering $Y$ such that $Y_1$ and $Y_2$ are subextensions of $Y$. If $\mathcal{Y}_1$ is a projective, integral, normal and flat model of $Y_1$ then $\widetilde{\mathcal{Y}_1}$, the normalization of $\mathcal{Y}_1$ is a model of $Y$ together with an morphism $\widetilde{\mathcal{Y}_1} \to \mathcal{Y}_1$. By the same reasoning we obtain a second model $\widetilde{\mathcal{Y}_2}$ of $Y$ with a morphism to $\mathcal{Y}_1$. By Lemma 4.15 the models $\widetilde{\mathcal{Y}_1}$ and $\widetilde{\mathcal{Y}_2}$ of $Y$ are dominated by a projective, integral, normal and flat model $\mathcal{Y}$ of $Y$. Now we can apply the same reasoning as in the case of abelian varieties to show that $\rho_1$ and $\rho_2$ are temperate for the covering $Y$ and the model $\mathcal{Y}$ □

We will attach now a vector bundle with connection to a temperate representation:

**Construction 4.20.** We proceed with the previous notations. Assume that the action of $G$ on $Y$ extends to $\mathcal{Y}$, and assume that every point $y$ of $\mathcal{Y}$ has an affine open neighborhood that is stable under the action of $G$. The last condition is satisfied if $\mathcal{Y}$ is quasi-projective.

Let $\rho : \pi_1^{alg}(X,x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ be an integral temperate representation. Fix some $n \geq 1$. Denote the kernel of the representation

$$\rho_n|_{\pi_1^{alg}(Y,y)} : \pi_1^{alg}(Y,y) \to Aut_{\mathfrak{o}}(\mathbb{L}_n)$$

by $N$. Then we have an exact sequence

$$0 \to \underbrace{\pi_1^{alg}(Y,y)/N}_{H:=} \to \underbrace{\pi_1^{alg}(X,x)/N}_{P:=} \to \underbrace{\pi_1^{alg}(X,x)/\pi_1^{alg}(Y,y)}_{=G} \to 0$$

Let $U := \mathrm{Spec}A \subset \mathcal{Y}$ be an affine open subset that is stable under the $G$-action. Let $\mathrm{Spec}B$ be the inverse image of $U$ under the finite étale Galois covering of $\mathcal{Y}$ corresponding to $H$. The action of $P$ on $B_{\overline{\mathbb{Q}_p}}$ extends to an action on $B$. This can be seen by the following reasoning. The ring $B$ is integral and normal, because $A$ has these properties ([Gro71] I Corollaire 9.10, Proposition 10.1). Moreover all elements of $B$ are integral over $A$ because $A = B^H$ ([Gro71] V Proposition 1.1 (i)). By the same reasoning all elements of $A$ are integral over $C := A^G$. Then $B$ is an integral normal ring whose elements are integral over $C$, in other words $B$ is the normalization of $C$ in the corresponding quotient fields whose Galois group is $P$. This implies that the action of $P$ on the quotient field of $B$ or on $B_{\overline{\mathbb{Q}_p}}$ extends to $B$. By definition the vector bundle $\mathcal{F}_{\rho_n|_H}$ attached to the representation $\rho_n|_H$ is defined on $U_{\mathfrak{o}}$ as the set of $H$ invariant sections:

$$\mathcal{F}_{\rho_n|_H}(U_{\mathfrak{o}}) = (B_{\mathfrak{o},n} \otimes_{\mathfrak{o}_n} \mathbb{L}_n)^H$$

where the action is the diagonal action. Also the group $P$ acts on $B_{\mathfrak{o},n} \otimes_{\mathfrak{o}_n} \mathbb{L}_n$ by the diagonal action and maps $H$-invariant sections to $H$-invariant sections:

$$h(pm) = ph'p^{-1}(pm) = ph'm = pm$$

for $m \in (B_{\mathfrak{o},n} \otimes_{\mathfrak{o}_n} \mathbb{L}_n)^H$, $p \in P$ $h, h' \in H$ ($N$ is a normal divisor in $P$). By the same reasoning one can show, that the corresponding connection $\nabla_{\rho_n|H}$ is equivariant under the $P$-action. As $N$ acts trivial, the quotient $G = P/N$ acts on $\mathcal{F}_{\rho_n|_H}(U_{\mathfrak{o}})$ and the action is equivariant for the connection. As this is true for all $n$, we get an $G$-action on the inverse limit

$$\varprojlim_n (\mathcal{F}_{\rho_n|_{\pi_1^{alg}(Y,y)}}, \nabla_{\rho_n|_{\pi_1^{alg}(Y,y)}})$$

and also on its generic fiber. Now we can use descent theory to obtain a vector bundle $(F_\rho, \nabla_\rho)$ with connection on $X_{\mathfrak{o}}$ attached to the temperate representation $\rho$. This follows by the same reasoning as in Section 4.1 or one can alternatively apply [Mum70] III Theorem 1 page 111. Let $\varphi$ be a morphism between two temperate representations. If $X$ is a curve or an abelian variety we can assume that both are temperate with the same covering $Y$ and model $\mathcal{Y}$ by Proposition 4.19. The morphism $\varphi$ induces a morphism between the corresponding vector bundles with connection. (by the same reasoning as in Sections 4.1 and 4.2)

The construction depends on the choice of a covering $Y \to X$ and the choice of a model $\mathcal{Y}$ of $Y$ with $G$-operation. We assume that one can check as in [DeWe05a] and [DeWe05b] that the construction is well defined (at least in the case of abelian varieties or curves). We will not do this but use an indirect argument: In Section 4.5 we will show that the construction is an inverse to the functor defined by Deninger-Werner. We will need to make the following assumption:

**Assumption 4.21.** The DeWe-functor is fully faithful.

If this assumption is true, then the construction is well defined by looking at the homomorphisms. So far the fully faithfulness of the DeWe-functor is only known for abelian varieties with good *ordinary* reduction [Wie06]. This implies also the fully faithfulness for line bundles on curves with good ordinary reduction by using the Jacobian embedding.

## 4.5 Vector bundles attached to representations and their relation to DeWe-representations

$X$ a smooth projective curve over $\overline{\mathbb{Q}_p}$
$x$ a $\overline{\mathbb{Q}_p}$-valued base point of $X$
$A$ an abelian variety over $\overline{\mathbb{Q}_p}$ having good reduction

In this section we want to show, that the assignment $\rho \to (\mathcal{F}_\rho, \nabla_\rho)$ of Section 4.4 defines an inverse functor to the construction of Deninger and Werner (Section 3.2) (for vector bundles attached to temperate representations).

We only study the case of vector bundles attached to temperate representations of the fundamental group of the curve $X$. The case of abelian varieties is similar and easier.

**Proposition 4.22.** *Let $\rho : \pi_1^{alg}(X, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ be a continuous temperate representation on a free $\mathfrak{o}$ -module $\mathbb{L}$ of rank $r$. Then the vector bundle $F_\rho$ lies in $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$ and the associated DeWe representation satisfies*

$$\rho^{DW}_{F_\rho} \cong \rho_{\mathbb{C}_p}$$

*i.e. the association $\rho \to (F_\rho, \nabla_\rho)$ defines an inverse functor to the construction of Deninger and Werner (for vector bundles attached to temperate representations).*

*Proof.* Let $\alpha : Y \to X$ be the corresponding étale Galois covering with group $G$, and let $y$ be a point above $x$. Let $\mathcal{Y}$ be a semi-stable, integral, normal, flat and projective model of $Y$ such that the $G$ action on $Y$ extends to $\mathcal{Y}$, and such that $\rho|_{\pi_1^{alg}(Y,y)}$ factors over $\pi_1^{alg}(\mathcal{Y}, y)$ (See Remark 4.18). Fix some $n \geq 1$. Denote the kernel of the representation

$$\rho_n|_{\pi_1^{alg}(Y,y)} : \pi_1^{alg}(Y, y) \twoheadrightarrow \pi_1^{alg}(\mathcal{Y}, y) \to Aut_{\mathfrak{o}}(\mathbb{L}_n)$$

by $N$. Then we have an exact sequence

$$0 \to \underbrace{\pi_1^{alg}(Y, y)/N}_{H:=} \to \underbrace{\pi_1^{alg}(X, x)/\alpha_* N}_{P:=} \to \underbrace{\pi_1^{alg}(X, x)/\pi_1^{alg}(Y, y)}_{=G} \to 0$$

Let $\pi : \mathcal{Z} \to \mathcal{Y}$ be the finite étale covering corresponding to $N$. Because $\mathcal{Y}$ is semi-stable $\mathcal{Z}$ is also semi-stable hence we have $\Gamma(\mathcal{Z}_{\mathfrak{o},n}, \mathcal{O}_{\mathcal{Z}_{\mathfrak{o},n}}) \cong \mathfrak{o}_n$ (see Remark 4.9). Fix a point $z$ in $\mathcal{Z}_{\overline{\mathbb{Q}_p}}$ lying over $y$.

Let $\mathcal{E}$ be the vector bundle on $\mathcal{Y}_{\mathfrak{o}}$ attached to the representation $\rho|_{\pi_1^{alg}(Y,y)}$ as in Section 4.4. Then $\mathcal{E}$ is a model of $\alpha^* F$.

Let $\gamma \in \pi_1^{alg}(X, x)$ be a path, and let $\gamma'$ be the unique path with $\alpha_*\gamma' = \gamma$ from $y$ to another point $y'$ over $x$. Consider the diagram:

$$\mathcal{E}_{y_n} \xleftarrow{z_n^*} \Gamma(\mathcal{Z}_n, \pi_n^* \mathcal{E}_n) \xrightarrow{(\gamma' z)_n^*} \mathcal{E}_{y'_n}.$$

By definition of the DeWe parallel transport (proof of Theorem 3.5)

$$\rho_{\mathcal{E},n}(\gamma') = (\gamma' z)_n^* \circ (z_n^*)^{-1}.$$

Let $\sigma \in P$ be the unique automorphism of $\mathcal{Z}_{\overline{\mathbb{Q}_p}}$ mapping $z$ to $\gamma z$. Then $\sigma$ extends uniquely to an automorphism of $\mathcal{Z}$ (mapping $z$ to $\gamma z$). We also have $\gamma' z = \sigma(z)$. By construction of the vector bundle $\mathcal{E}_\rho$ the module $\Gamma(\mathcal{Z}_n, \pi_n^* \mathcal{E}_n)$ is isomorphic to $\mathbb{L}_n \otimes_{\mathfrak{o}} \Gamma(\mathcal{Z}_n, \mathcal{O}_{\mathcal{Z}_n}) \cong \mathbb{L}_n \otimes_{\mathfrak{o}} \mathfrak{o}_n$ as a $P$-module. Therefore for $v \otimes f \in \mathbb{L}_n \otimes_{\mathfrak{o}} \Gamma(\mathcal{Z}_n, \mathcal{O}_{\mathcal{Z}_n})$

$$(\gamma' z)_n^*(v \otimes f) = (\sigma z)_n^*(v \otimes f) = z_n^* \circ \sigma^*(v \otimes f) = z_n^*(\rho(\sigma) m \otimes \sigma^* f)$$

As $\sigma^* f = f$ ($f$ is constant) we have the following relation:

$$\rho_{\mathcal{E},n}(\gamma') = \gamma'(z)_n^* \circ (z_n^*)^{-1} = z_n^* \circ \rho_n(\sigma) \circ (z_n^*)^{-1}.$$

By construction of the DeWe representations (proof of Theorem 3.5) we have

$$(\rho_{F_\rho})_n(\gamma) = \rho_{\mathcal{E}_n}(\gamma')$$

This implies that the projective limits of the representations $(\rho_{F_\rho})_n$ and $\rho_n$ are isomorphic, i.e.

$$\rho_{F_\rho}^{DW} \cong \rho$$

□

# 5 A comparison between the algebraic and the topological Riemann-Hilbert correspondence

| | |
|---|---|
| $X/\overline{\mathbb{Q}_p}$ | a proper smooth curve over $\overline{\mathbb{Q}_p}$ |
| $x$ | a base point of $X$ |
| $X^{an}_{\mathbb{C}_p}$ | the analytification of $X_{\mathbb{C}_p}$ |
| $\Gamma := \pi_1^{top}(X^{an}_{\mathbb{C}_p}, x)$ | the topological fundamental group of $X^{an}_{\mathbb{C}_p}$ |
| $\pi_1^{alg}(X, x)$ | the algebraic fundamental group of $X$ |

## 5.1 A comparison between the algebraic and the topological Riemann-Hilbert correspondence

| | |
|---|---|
| $\mathbb{L}$ | a free rank $r$ module over $\mathfrak{o}$ |
| $\rho$ | a representation $\pi_1^{top}(X^{an}_{\mathbb{C}_p}, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ |

In this section we will relate the topological and the algebraic approach to the Riemann-Hilbert correspondence. This was already done in the case of Mumford curves omitting connections by G. Herz (Section 3.11) in his dissertation. Our comparison is a generalization of his work to arbitrary curves, vector bundles equipped with connections and works over $\mathbb{C}_p$.

Let $\mathcal{X}^{nsc}$ be a projective integral normal flat $\overline{\mathbb{Z}_p}$ model of $X$ with non-singular components as defined in Section 3.8. The formal completion $\hat{\mathcal{X}}^{nsc}_{\mathfrak{o}}$ of $\mathcal{X}^{nsc}_{\mathfrak{o}}$ has an universal covering $u : \hat{\Omega}^{\circ}_{\mathfrak{o}} \to \hat{\mathcal{X}}^{nsc}_{\mathfrak{o}}$ whose (analytic) generic fiber is the universal covering of $X^{an}_{\mathbb{C}_p}$. The group $\pi_1^{ftop}(X_{\mathbb{C}_p}, x)$ (Section 3.6) classifies finite topological covers of $X^{an}_{\mathbb{C}_p}$ and is the the pro-finite completion of $\Gamma := \pi_1^{top}(X^{an}_{\mathbb{C}_p}, x)$ ([Her05] Remark 1.4.1 (7)). Let $\pi_1^{alg}(\mathcal{X}^{nsc}, x)$ be the algebraic fundamental group of the model $\mathcal{X}^{nsc}$. The morphism $X \to \mathcal{X}^{nsc}$ induces a morphism

$$\pi_1^{alg}(X, x) \to \pi_1^{alg}(\mathcal{X}^{nsc}, x).$$

The representation $\rho$ induces a continuous representation $\hat{\rho}$ of $\pi_1^{ftop}(X^{an}_{\mathbb{C}_p}, x)$, the pro-finite completion of $\rho$. The representation $\hat{\rho}$ induces a continuous representation $\psi$ of $\pi_1^{alg}(\mathcal{X}^{nsc}, x)$. This can be seen as follows: The reduction modulo $p^n$ of $\hat{\rho}$ factors over a finite group $G_n$ because $\pi_1^{ftop}(X^{an}_{\mathbb{C}_p}, x)$ has the pro-finite topology and $Aut_{\mathfrak{o}}(\mathbb{L})$ the discrete one. To this group $G_n$ there corresponds a finite topological Galois covering $Y \to X^{an}_{\mathbb{C}_p}$. By the universal property of the universal covering $\Omega^{an}_{\mathbb{C}_p}$ of $X_{\mathbb{C}_p}$ there exist a normal subgroup $N \subset \Gamma$ with $G_n \cong \Gamma/N$ and $\Omega^{an}_{\mathbb{C}_p}/N \cong Y$. The universal covering $\Omega^{an}_{\mathbb{C}_p}$ is the generic fiber of the formal scheme $\hat{\Omega}^{\circ}_{\mathfrak{o}}$ and $Y$ is the generic fiber of the formal scheme $\hat{\Omega}_{\mathfrak{o}}^{\circ}/N$. We also have $(\hat{\Omega}_{\mathfrak{o}}^{\circ}/N)/G \cong \hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}$ and $\hat{\Omega}_{\mathfrak{o}}^{\circ}/N$ is a finite topological covering of $\hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}$ in the Zariski (hence étale) topology. We note that $\hat{\Omega}_{\mathfrak{o}}^{\circ}$ and all its quotients are defined over the ring of integers of a finite extension of $\mathbb{Q}_p$, because this is true for $\mathcal{X}_{\mathfrak{o}}^{nsc}$. By [EGA] III (premiere partie) Proposition 5.4.4 the formal scheme $\hat{\Omega}_{\mathfrak{o}}^{\circ}/N$ is the formal completion of an algebraic model $\mathcal{Y}$ (after base change with $\mathfrak{o}$) which is a finite étale Galois cover of $\mathcal{X}_{\mathfrak{o}}^{nsc}$ with group $G_n$. Hence the topological covering $Y \to X_{\mathbb{C}_p}^{an}$ is the generic fiber of the finite étale Galois covering $\mathcal{Y}_{\mathfrak{o}} \to \mathcal{X}_{\mathfrak{o}}^{nsc}$. We can then define

$$\psi_n : \pi_1^{alg}(\mathcal{X}^{nsc}, x) \to G_n \to Aut_{\mathfrak{o}_n}(\mathbb{L}_n).$$

The projective limit over these representations defines a representation

$$\psi : \pi_1^{alg}(\mathcal{X}^{nsc}, x) \to Aut_{\mathfrak{o}}(\mathbb{L}).$$

We can now compare the algebraic and the topological Riemann-Hilbert correspondence:

**Theorem 5.1.** *The formal vector bundle with connection $RH^{top,\circ}(\rho) = (M_\rho^\circ, \nabla_\rho^\circ)$ attached to the representation $\rho$ of $\pi_1^{top}(X_{\mathbb{C}_p}^{an}, x)$ is isomorphic to $(\hat{\mathcal{F}}_\psi, \hat{\nabla}_\psi)$, the formal completion of the algebraic vector bundle attached to the representation $\psi$ of $\pi_1^{alg}(\mathcal{X}_{\mathfrak{o}}^{nsc}, x)$, i.e.*

$$(M_\rho^\circ, \nabla_\rho^\circ) \cong (\hat{\mathcal{F}}_\psi, \hat{\nabla}_\psi) \qquad \text{on } \hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}.$$

*The same is true for the generic fibers, i.e.*

$$(M_\rho, \nabla_\rho) \cong (\hat{\mathcal{F}}_\psi \otimes_{\mathfrak{o}} \mathbb{C}_p, \hat{\nabla}_\psi \otimes_{\mathfrak{o}} \mathbb{C}_p) \qquad \text{on } X_{\mathbb{C}_p}^{an}.$$

*Proof.* It suffices to compare both constructions on the model $\mathcal{X}_{\mathfrak{o}}^{nsc}$ modulo $p^n$. We write $\rho_n$ and $\psi_n$ for the reduction modulo $p^n$. The representation $\psi_n$ factors over a finite group $G$. Let $\varphi : \mathcal{Y} \to \mathcal{X}^{nsc}$ be the corresponding finite étale $G$-cover and write $\hat{\varphi}$ for the corresponding map of formal schemes. The analytic generic fiber of the covering $\varphi : \mathcal{Y}_{\mathfrak{o}} \to \mathcal{X}_{\mathfrak{o}}^{nsc}$ is a finite topological Galois covering with the same group $G$, and $\rho_n$ factors over this group. Denote by $N$ and $\hat{N}$ the kernels of $\rho_n$ and $\psi_n$. Note, that $\pi_1^{alg}(\mathcal{X}^{nsc}, x)/\hat{N} \cong G \cong \Gamma/N$.

We want to calculate the reduction modulo $p^n$ of the formal vector bundle $(M_\rho^\circ, \nabla_\rho^\circ)$ attached to $\rho$. Let $U \subset \hat{\mathcal{X}}_{\mathfrak{o}}^{nsc}$ be an open subset, then

$$M_\rho^\circ(U) = \{m \in \mathbb{L} \otimes_{\mathfrak{o}} \mathcal{O}_{\hat{\Omega}_{\mathfrak{o}}^\circ}(u^{-1}U) | \gamma(m) = m \text{ for all } \gamma \in \Gamma\}.$$

If we restrict the representation $\rho$ to $N \subset \Gamma$ (its kernel mod $p^n$), then

$$M_{\rho|_N}^\circ(\hat{\varphi}^{-1}U) = \{m \in \mathbb{L} \otimes_{\mathfrak{o}} \mathcal{O}_{\hat{\Omega}_{\mathfrak{o}}^\circ}(u^{-1}U) | \gamma(m) = m \text{ for all } \gamma \in N\}. \qquad (10)$$

The reduction modulo $p^n$ of the vector bundle $M^\circ_{\rho|_N}$ attached to $\rho|_N$ on $\hat{\mathcal{Y}}_{\mathfrak{o}}$ is isomorphic to the trivial vector bundle $\mathcal{O}_{\hat{\mathcal{Y}}_{\mathfrak{o}},n} \otimes_{\mathfrak{o}_n} \mathbb{L}_n$ because $\rho_n|_N \equiv 1$, and hence $\rho_n$ acts trivially on $\mathbb{L}_n$ in (10). This implies, that

$$(M^\circ_\rho(U))_n = (((\mathbb{L} \otimes_{\mathfrak{o}} \mathcal{O}_{\hat{\Omega}^\circ_{\mathfrak{o}}}(u^{-1}U))^N)^{\Gamma/N})_n =$$

$$= ((M^\circ_{\rho|_N}(\hat{\varphi}^{-1}U))_n)^G = (\mathcal{O}_{\hat{\mathcal{Y}}_{\mathfrak{o}},n}(\hat{\varphi}^{-1}U) \otimes_{\mathfrak{o}_n} \mathbb{L}_n)^G,$$

and we recover the construction of the vector bundle $\mathcal{F}_{\psi_n}$ attached to the representation $\psi_n : G \to Aut_{\mathfrak{o}_n}(\mathbb{L}_n)$ as in Section 4.1. But this is the reduction modulo $p^n$ of the vector bundle $\mathcal{F}_\psi$.

It remains to calculate the reduction modulo $p^n$ of the connection of $RH^{top,\circ}(\rho)$.

$$\nabla^\circ_\rho(\sum_{i=1}^r e_i \otimes f_i) = \sum_{i=1}^r e_i \otimes df_i$$

for $U \subset \hat{\mathcal{X}}^{nsc}_{\mathfrak{o}}$ open, $e_i$ a basis of $\mathbb{L}$, $f_i \in \mathcal{O}_{\hat{\Omega}^\circ_{\mathfrak{o}}}(u^{-1}U)$.

As calculated above

$$(M^\circ_\rho(U))_n = ((M^\circ_{\rho|_N}(\hat{\varphi}^{-1}U))_n)^G \cong (\mathcal{O}_{\hat{\mathcal{Y}}_{\mathfrak{o}},n}(\hat{\varphi}^{-1}U) \otimes_{\mathfrak{o}_n} \mathbb{L}_n)^G$$

and $\nabla^\circ_{\rho|_N,n}$ is the "constant" connection on the trivial bundle $M^\circ_{\rho|_N,n}(U)$. This connection is $G$ equivariant and descends to a connection on $(M^\circ_\rho)_n$ that is isomorphic to $(\nabla^\circ_\rho)_n$ and $(\nabla_\psi)_n$ □

## 5.2 Comparison between Berkovich and DeWe parallel transport

In this section we will compare the parallel transport of Berkovich and Denigner-Werner using the comparison in Section 5.1.

**Definition 5.2.** Let $\mathfrak{B}^{top-rep}_{X^{an}_{\mathbb{C}_p}}$ be the category of vector bundles with connection on $X_{\mathbb{C}_p}$, that are attached to a representation $\rho : \pi_1^{top}(X^{an}_{\mathbb{C}_p}, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ on finitely generated free $\mathfrak{o}$-modules $\mathbb{L}$. This is also the image of Andrés' topological Riemann-Hilbert functor $RH^{top}$ (for integral representations).

The vector bundles with connections in $\mathfrak{B}^{top-rep}_{X^{an}_{\mathbb{C}_p}}$ satisfy the following properties:

**Proposition 5.3.** *Let $(M_\rho, \nabla_\rho) \in \mathfrak{B}^{top-rep}_{X^{an}_{\mathbb{C}_p}}$ be a vector bundle attached to an integral representation $\rho$ of $\pi_1^{top}(X^{an}_{\mathbb{C}_p}, x)$ on $Aut_{\mathfrak{o}}(\mathbb{L})$. Then* *a)* $(M_\rho, \nabla_\rho)$ *is a quasi-unipotent* $\mathcal{D}_X$*-module of level* 1 *at each point* $x \in X^{an}_{\mathbb{C}_p}$*;*

*b) Every point* $x \in X^{an}_{\mathbb{C}_p}$ *has an étale neighborhood* $U \to X^{an}_{\mathbb{C}_p}$ *such that there is an embedding (even an isomorphism) of* $\mathcal{D}_U$*-modules*

$$M_\rho|_U \hookrightarrow (\mathcal{S}^{\lambda,0}_U)^1 = \mathcal{O}^r_U \ ;$$

*c) The sheaf* $(M_\rho)^{\nabla_\rho}_{\mathcal{S}^\lambda_{X^{an}_{\mathbb{C}_p}}}$ *of horizontal sections is equal to* $M_\rho^{\nabla_\rho}$*, the locally constant sheaf on* $X^{an}_{\mathbb{C}_p}$ *attached to the representation* $\rho$*;*

*d) The representations of* $\pi_1^{top}(X^{an}_{\mathbb{C}_p}, x)$ *attached to* $(M_\rho, \nabla_\rho)$ *by André and Berkovich are both isomorphic to* $\rho_{\mathbb{C}_p}$*, the extension of* $\rho$ *to* $\mathbb{L} \otimes_{\mathfrak{o}} \mathbb{C}_p$*.*

*Proof.* The pullback of $(M_\rho, \nabla_\rho)$ under the universal covering map $u : \Omega^{an}_{\mathbb{C}_p} \to X^{an}_{\mathbb{C}_p}$ is by construction the trivial bundle on $\Omega^{an}_{\mathbb{C}_p}$. Topological coverings are (Berkovich) étale, so we can take $U = \Omega^{an}_{\mathbb{C}_p}$ in b). Hence a),b) follow. c) follows from the construction of the sheaf $(M_\rho, \nabla_\rho)$ and b). Finally d) follows from c) and the fact that $M_\rho^{\nabla_\rho}$ is the locally constant sheaf $\mathcal{V}_{\rho_{\mathbb{C}_p}}$ corresponding to the representation $\rho_{\mathbb{C}_p}$ in Andrés Riemann-Hilbert correspondence □

We can now summarize our results in

**Theorem 5.4.** *Let* $(M_\rho, \nabla_\rho) \in \mathfrak{B}^{top-rep}_{X^{an}_{\mathbb{C}_p}}$ *be a vector bundle with connection attached to a (discrete) representation* $\rho : \pi_1^{top}(X^{an}_{\mathbb{C}_p}, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$*, where* $\mathbb{L}$ *is a free* $\mathfrak{o}$*-module of rank* $r$*. Let* $\hat{\rho}$ *be the pro-finite completion of* $\rho$*, and let* $\psi$ *be the induced representation of* $\pi_1^{alg}(\mathcal{X}, x)$ *and* $\pi_1^{alg}(X, x)$ *(see Section 5.1). Then*

*a)* $(M_\rho, \nabla_\rho)$ *is the analytification of the algebraic vector bundle with connection* $(\mathcal{F}_\psi \otimes_{\mathfrak{o}} \mathbb{C}_p, \nabla_\psi \otimes_{\mathfrak{o}} \mathbb{C}_p)$ *on* $X_{\mathbb{C}_p}$ *attached to the representation* $\psi$*;*

*b) The vector bundle* $F_\psi := \mathcal{F}_\psi \otimes_{\mathfrak{o}} \mathbb{C}_p$ *lies in* $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$*, the category defined by Deninger and Werner;*

*c) The corresponding DeWe-representation* $\rho^{DeWe}_{F_\psi}$ *is defined and isomorphic to* $\psi_{\mathbb{C}_p}$*, where* $\psi_{\mathbb{C}_p}$ *is the extension of* $\psi : \pi_1^{alg}(X, x) \to Aut_{\mathfrak{o}}(\mathbb{L})$ *to* $Aut_{\mathbb{C}_p}(\mathbb{L} \otimes_{\mathfrak{o}} \mathbb{C}_p)$*;*

*d)* $(M_\rho, \nabla_\rho)$ *is étale locally unipotent, and Berkovich parallel transport is defined;*

*e) Let* $\rho^{Ber} : \pi_1^{top}(X^{an}_{\mathbb{C}_p}, x) \to Aut_{\mathbb{C}_p}((M_\rho)_x^{\nabla_\rho})$ *be the representation obtained by Berkovich parallel transport along paths from* $x$ *to* $x$ *of elements of the fiber* $(M_\rho)_x^{\nabla_\rho}$ *over* $x$ *of horizontal sections of* $\nabla_\rho$. *Then* $\rho^{Ber}$ *is isomorphic to* $\rho_{\mathbb{C}_p}$, *the extension of* $\rho$ *to* $Aut_{\mathbb{C}_p}(\mathbb{L} \otimes_{\mathfrak{o}} \mathbb{C}_p)$.

*Hence for all vector bundles in* $\mathfrak{B}^{top-rep}_{X^{an}_{\mathbb{C}_p}}$ *the parallel transports of André, Berkovich, Faltings, van der Put-Reversat (topological) and Deninger-Werner, Faltings (algebraic) are compatible.*

**Remark 5.5.** We assume that this correspondence can be extended vector bundles with connection attached to representations of $\pi_1^{temp}(X^{an}_{\mathbb{C}_p}, x)$, i.e. to vector bundles with connection that lie in the image of $RH^{temp}$ (Andrés' temperate Riemann-Hilbert functor).

# 6 Homogeneous vector bundles on abelian varieties

| | |
|---|---|
| $A$ | an abelian variety over $\overline{\mathbb{Q}_p}$ with good reduction |
| $\mathcal{A}$ | an abelian scheme over $\overline{\mathbb{Z}_p}$ with generic fiber $A$ |
| $x : \mathrm{Spec}\overline{\mathbb{Q}_p} \to A$ | the zero section of $A$ |

In this section we will show that the category $\mathfrak{B}_{A_{\mathbb{C}_p}}$ defined by Deninger Werner in Section 3.3 consists of the homogeneous (translation invariant) vector bundles on $A_{\mathbb{C}_p}$ under the assumption that the DeWe functor $\rho^{DeWe}$ is fully faithful.

**Definition 6.1.** A vector bundle $F$ on $A_{\mathbb{C}_p}$ is called homogeneous (or translation invariant) if $T_a^* F \cong F$ for all $a \in A(\mathbb{C}_p)$, where $T_a$ denotes the translation by $a$ map.

The following theorem of Matsushima and Morimoto classifies homogeneous vector bundles on complex tori:

**Theorem 6.2 (Matsushima, Morimoto).** *Let $S$ be a complex torus, and let $F$ be a vector bundle on $S$. Then the following are equivalent:*

*a) $F$ has a connection.*

*b) $T_a^* F \cong F$ for all $a$ in $S$ ($F$ is homogeneous).*

*c) $F$ has an integrable connection.*

*d) Each indecomposable component is uniquely of the form*

$$L \otimes U$$

*where $L$ is a line bundle of degree zero, and $U$ is a unipotent vector bundle, i.e. a successive extension of the trivial line bundle.*

*Proof.* See [Oda71] for references □

**Proposition 6.3.** *Let $F \in \mathfrak{B}_{A_{\mathbb{C}_p}}$ be a vector bundle, and let $a \in A(\mathbb{C}_p)$ be a point. Then the DeWe-representations $\rho_F$ and $\rho_{T_a^* F}$ attached to $F$ and $T_a^* F$ are isomorphic.*

*Proof.* By definition $F$ has a model $\mathcal{F}$ on $\mathcal{A}_{\mathfrak{o}}$, and for every $n \geq 1$ there exists a $N \geq 1$ such that $(N^*\mathcal{F})_n$ is isomorphic to the trivial vector bundle on $\mathcal{A}_n$. The vector bundle $T_a^* F$ has the model $T_a^* \mathcal{F}$ on $\mathcal{A}_{\mathfrak{o}}$ and $(N^* T_a^* \mathcal{F})_n$ is isomorphic to the trivial vector bundle on $\mathcal{A}_n$. This can be seen as follows: Let

$b \in A(\mathbb{C}_p)$ be a point satisfying $N \cdot b = a$. Then $N \circ T_b = T_a \circ N$, hence there is a canonical isomorphism $(T_b^* N^* \mathcal{F})_n \cong (N^* T_a^* \mathcal{F})_n$ and $(N^* T_a^* \mathcal{F})_n$ is isomorphic to the trivial vector bundle. We can now compute and compare the representations $\rho_F$ and $\rho_{T_a^* F}$ modulo $p^n$: Consider the following commutative diagram:

$$\begin{array}{ccccc}
\Gamma(\mathcal{A}_n, (N^*\mathcal{F})_n) & \xrightarrow[\sim]{x_n^*} & \Gamma(\mathrm{Spec}\,\mathfrak{o}_n, x_n^* \mathcal{F}_n) & = & \mathcal{F}_{x_n} \\
T_b^* \downarrow \wr & & & & \\
\Gamma(\mathcal{A}_n, (T_b^* N^* \mathcal{F})_n) & & & & T_a^* \downarrow \wr \\
can \downarrow \wr & & & & \\
\Gamma(\mathcal{A}_n, (N^* T_a^* \mathcal{F})_n) & \xrightarrow[\sim]{x_n^*} & \Gamma(\mathrm{Spec}\,\mathfrak{o}_n, x_n^* (T_a^* \mathcal{F})_n) & = & (T_a^* \mathcal{F})_{x_n}
\end{array}$$

Note that the composition $\mathrm{can} \circ T_b^*$ on the left hand side does not depend on the choice of the point $b$, because the other three morphisms $x_n^*$, $x_n^*$ and $T_a^*$ do not depend on $b$. The group $A_N(\overline{\mathbb{Q}_p})$ of torsion points acts on the modules $\Gamma(\mathcal{A}_n, (N^*\mathcal{F})_n)$, $\Gamma(\mathcal{A}_n, (T_b^* N^* \mathcal{F})_n)$ and $\Gamma(\mathcal{A}_n, (N^* T_a^* \mathcal{F})_n)$ by translation. The isomorphisms on the left hand side $T_b^*$ and *can* are equivariant under the action of $A_N(\overline{\mathbb{Q}_p})$, because $T_b^* T_y^* \cong T_y^* T_b^*$ for all $y \in A_N(\overline{\mathbb{Q}_p})$. This implies that the representations $\rho_{F,n}$ and $\rho_{T_a^* F, n}$ are isomorphic. The same holds for the $\rho_F$ and $\rho_{T_a^* F}$ by taking limits □

**Proposition 6.4.** *Assume that the DeWe-functor $\rho^{DeWe}$ is fully faithful. Then a vector bundle lies in $\mathfrak{B}_{A_{\mathbb{C}_p}}$ if and only if it is a homogeneous vector bundle.*

*Proof.* If a vector bundle is homogeneous, then by Theorem 6.2 it is the tensor product of an unipotent vector bundle and a line bundle of degree zero (We use the fact that there exists an isomorphism $\mathbb{C} \cong \mathbb{C}_p$ [Rob00] Section 3.5 page 144). Any such vector bundles lie in $\mathfrak{B}_{A_{\mathbb{C}_p}}$ by Theorem 3.7. If $F$ is a vector bundle in $\mathfrak{B}_{A_{\mathbb{C}_p}}$, and $a \in A(\mathbb{C}_p)$ is a point, then by Proposition 6.3 the DeWe-representations $\rho_F^{DeWe}$ and $\rho_{T_a^* F}^{DeWe}$ attached to the vector bundles $F$ and $T_a^* F$ are isomorphic. Under the assumption, that the DeWe-functor is fully faithful we obtain an isomorphism between $F$ and $T_a^* F$ and so $F$ is homogeneous □

**Remark 6.5.** For an abelian variety $A$ with totally degenerate reduction, it was shown by van der Put - Reverasat [PuRe88], that homogeneous vector bundles correspond to "$\Phi$-bounded" representations on finite dimensional $\mathbb{C}_p$-vector spaces, of the topological fundamental group of $A$, using the methods of Faltings [Fal83]. We assume that after developing the theory of Deninger and Werner to abelian varieties with bad reduction, one can compare the algebraic approach and the topological as for curves.

# 7 Applications and remarks

## 7.1 Representations attached to vector bundles on abelian varieties

| | |
|---|---|
| $K$ | a finite extension of $\mathbb{Q}_p$ |
| $V$ | its ring of integers |
| $k$ | its residue field |
| $\mathcal{A}$ | an abelian scheme over $V$ of dimension $d$ |
| $A$ | its generic fiber |

In this section we characterize the DeWe-representations that are attached to vector bundles in $\mathfrak{B}_{A_{\mathbb{C}_p}}$. These were already classified in the case of line bundles of degree 0 [DeWe05a] and in the case of unipotent vector bundles if $A$ has good ordinary reduction and is a canonical lift [Wie06] Chapter 4.

**Remark 7.1.** Let

$$\rho : TA \to GL_r(\mathfrak{o})$$

be a continuous representation. Let $U \subset GL_r(\mathfrak{o})$ be the open subgroup of matrices reducing to the identity $E_r$ modulo $p^{1/(p-1)}$. As $\rho$ is continuous, the group $P := \rho^{-1}(U) \subset TA$ is an open subgroup. Hence $\rho$ restricted to $P$ maps into $U$. The logarithm series converges for arguments divisible by $p^{1/(p-1)}$. Hence we obtain a map

$$P \xrightarrow{\rho} U \xrightarrow{\log} M_r(\mathbb{C}_p)$$

We will use Hodge-Tate theory to investigate this map:

**Proposition 7.2 (Tate).** *Let $G = Gal(\overline{\mathbb{Q}_p}/K)$ be the absolute Galois group of $K$. Then*

$$H^0(G_K, \mathbb{C}_p) = K$$

*and for $n \neq 0$*

$$H^0(G_K, \mathbb{C}_p(n)) = 0$$

*Proof.* [Tate66] Theorem 1, Theorem 2 □

**Proposition 7.3 (Tate).** *There is a Hodge-Tate ($\mathcal{C}_{HT}$) decomposition (of Galois modules)*

$$Hom_{\mathbb{Z}_p}(T_pA, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong H^1_{et}(A_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \mathbb{C}_p(-1)^{dimA} \oplus \mathbb{C}_p^{dimA}$$

*Proof.* [Tate66] Corollary 2 □

**Corollary 7.4.** *The following holds:*

$$dim_K Hom_{c,G_K}(TA, \mathbb{C}_p) = dim A.$$

*Proof.* Use the fact that $Hom_c(TA, \mathbb{C}_p) = Hom_{\mathbb{Z}_p}(T_pA, \mathbb{C}_p)$ and Hodge-Tate theory □

**Corollary 7.5.** *If $A$ has (good) ordinary reduction, then the map*

$$TA = \pi_1^{alg}(A_{\overline{K}}, 0) \twoheadrightarrow \pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0)$$

*induces an isomorphism*

$$Hom_{c,G_K}(\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0), \mathbb{C}_p) \cong Hom_{c,G_K}(TA, \mathbb{C}_p).$$

*Proof.* The left hand side is a $K$ sub-vector-space of the right hand side. It suffices to show, that its dimension is $dim A$. Let $\mathcal{A}^{et}[p^\infty]$ be the étale part of the connected-étale sequence corresponding to the $p$-Barsotti-Tate group $\mathcal{A}[p^\infty]$ associated to the $p$ torsion points of $\mathcal{A}$. Because $A$ has ordinary reduction we have

$$T\mathcal{A}^{et}[p^\infty] \cong (\lim_{\overleftarrow{n}} \mathbb{Z}/p^n)^d = (\mathbb{Z}_p)^d.$$

The claim follows as

$$Hom_c(\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0), \mathbb{C}_p) = Hom_c(T\mathcal{A}^{et}[p^\infty], \mathbb{C}_p)$$

□

**Proposition 7.6.** *Assume that $A$ has (good) ordinary reduction and let $\rho : TA \to GL_r(\mathfrak{o})$ be a continuous Galois-invariant representation. Then there exists some $N \geq 1$ such that the composition*

$$\psi : TA \xrightarrow{TN} TA \xrightarrow{\rho} GL_r(\mathfrak{o})$$

*factors over $\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0)$, i.e. it is a temperate representation. Here $TN$ is the map induced by $N$-multiplication on $A$.*

*Proof.* Choose a $N \geq 1$ such that $TN(TA) \subset P := \rho^{-1}(U)$. Then the logarithm series converges for elements in $U$ and is injective on $U$ (exp is the inverse). Consider the composite map

$$\psi : TA \xrightarrow{TN} TA \xrightarrow{\rho} GL_r(\mathfrak{o}).$$

By Corollary 7.5 the map

$$\log \circ \psi : TA \to M_r(\mathbb{C}_p)$$

factors over $\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0)$, because each component does. Because log restricted to $im\psi$ is injective, the map $\psi$ also factors over $\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0)$ □

**Remark 7.7.** We assume that a similar description of DeWe representations is possible if $A$ has non-ordinary (good) reduction. Let $M \subset TA$ be the intersection of all kernels of elements in $Hom_{c,G_K}(TA, \mathbb{C}_p)$. If $\rho$ is attached to vector bundle on $A$, then there exists a $N \geq 1$ such that $\rho \circ TN$ factors over $TA/M$. This is analogous to the definition of temperate representations and $TA/M$ plays the role of $\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0)$. However we do not know wether to any such representation one can attach a vector bundle, as in the ordinary case.

**Corollary 7.8.** *Let $E$ be a vector bundle in $\mathfrak{B}_{A_{\mathbb{C}_p}}$, which is defined over $K$. Then it is associated to a temperate representation of $TA = \pi_1^{alg}(A_{\overline{\mathbb{Q}_p}}, 0)$. The same holds if the vector bundle is defined over a finite extension $K'$ of $K$.*

*Proof.* This follows from Proposition 7.6 □

To show the same result for vector bundles defined over $\mathbb{C}_p$ we need to approximate vector bundles $p$-adically by a family of vector bundles defined over finite extensions of $K$.

We need to make the following assumption:

**Assumption 7.9.** Let $K'$ be a finite extension of $K$ with ring of integers $V'$. Let $\mathcal{E}$ be a vector bundle on $\mathcal{A}_{\mathfrak{o}}$, and let $\mathcal{E}^{V'_n}$ be a vector bundle on $\mathcal{A}_{V'_n}$ ($V'_n = V'/p^n$). Assume that $\mathcal{E}^{V'_n} \otimes_{V'_n} \mathfrak{o}_n \cong \mathcal{E}_n$. Then $\mathcal{E}^{V'_n}$ can be lifted to a vector bundle $\mathcal{E}^{V'}$ on $\mathcal{A}_{V'}$.

**Remark 7.10.** The assumption is satisfied is $A$ is an elliptic curve ([Her05] Lemma 2.20) because the obstruction lies in the second cohomology group that vanishes for curves. If $\mathcal{E}$ is a line bundle on $\mathcal{A}_{\mathfrak{o}}$ then it corresponds to a $\mathfrak{o}$-valued point on the dual abelian scheme $\hat{\mathcal{A}}_{\mathfrak{o}}$. Because $\overline{\mathbb{Z}_p}/p^n = \mathfrak{o}/p^n$ we see that $\hat{\mathcal{A}}(\overline{\mathbb{Z}_p}/p^n) = \hat{\mathcal{A}}(\mathfrak{o}/p^n)$, and the assumption is true also in the case of line bundles.

**Corollary 7.11.** *Let $E$ be a vector bundle in $\mathfrak{B}_{A_{\mathbb{C}_p}}$, which is defined over $\mathbb{C}_p$ and assume that Assumption 7.9 holds. Then it is associated to a temperate representation of $\pi_1^{alg}(A_{\overline{\mathbb{Q}_p}}, 0)$.*

*Proof.* Choose a $N \geq 1$ such that $TN(TA) \subset P := \rho^{-1}(U)$ (Notation as in Remark 7.1). We claim that the map $\psi := \rho \circ TN$ factors over $\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0)$. It suffices to show that $\psi_n = \rho_n \circ TN$ satisfies this property for every $n \geq 1$. Let $\mathcal{E}$ be a vector bundle on $\mathcal{A}_{\mathfrak{o}}$ with generic fiber $E$. Because $\mathfrak{o}_n = \overline{\mathbb{Z}_p}/p^n$, the vector bundle $\mathcal{E}_n$ is already defined over $V'/p^n$ for some finite extension $K'$ of $K$ with ring of integers $V'$. Assume that there exists a vector bundle

$\mathcal{E}^{V'}$ on $\mathcal{A}_{V'}$ defined over $V'$, and satisfying $\mathcal{E}^{V'} \otimes_{V'} \mathfrak{o}_n \cong \mathcal{E}_n$ (Assumption 7.9). Then the representations $\rho$ and $\rho'$ attached to the vector bundles $\mathcal{E}^{V'} \otimes_{V'} \mathfrak{o}$ and $\mathcal{E}$ are isomorphic modulo $p^n$. By Corollary 7.8 (with $K$ replaced by $K'$), the representation $\psi' := \rho' \circ TN$ corresponding to $\rho'$ factors over $\pi_1^{alg}(\mathcal{A}_{\overline{\mathbb{Z}_p}}, 0)$. Because $\psi$ and $\psi'$ are isomorphic modulo $p^n$, the claim follows □

## 7.2 A relation between Φ-bounded representations and representations of the algebraic fundamental group

$K$ a finite extension of $\mathbb{Q}_p$
$V$ its ring of integers

Let $X/K$ be a Mumford curve and let $\rho : \Gamma = \pi_1^{top}(X, x) \to GL_r(K)$ be a Φ-bounded representation ($x \in X$ is a geometric base point). It was shown by Faltings (Theorem 3.50) that Φ -bounded representations correspond to semi-stable vector bundles of degree 0 on $X$. It was shown by G. Herz (see Section 3.11), that if this representation is integral, i.e. it is a representation into $GL_r(V)$, then the vector bundle $M_\rho$ attached to $\rho$ lies in $\mathfrak{B}^s_X$, the category defined by Deninger-Werner, and the attached DeWe-representation is the pro-finite completion of $\rho$.

We will examine now the case of arbitrary Φ-bounded representations, i.e. Φ-bounded representations that are not necessarily integral. We will restrict us the case that $X$ is a Tate elliptic curve:

We will need a criterion to decide wether two vector bundles attached to two different representations are isomorphic:

**Lemma 7.12.** *Let $X$ be a smooth proper curve over $\mathbb{C}_p$, and denote by $X^{an}$ its analyticification. Let $\tilde{X}/X^{an}$ be an (Berkovich-) étale Galois covering with group $G$ (e.g topological or finite étale). Let $\rho_1, \rho_2 : G \to GL_r(\mathfrak{o})$ be two continuous (discrete topology) representations and denote by $M_{\rho_1}$, $M_{\rho_2}$ the attached vector bundles. Then the following two conditions are equivalent:*

*a) $M_{\rho_1} \cong M_{\rho_2}$;*

*b) There is an (Berkovich-) analytic function $f : \tilde{X} \to GL_r(\mathbb{C}_p)$ such that*

$$f(\gamma z) = \rho_2(\gamma) f(z) \rho_1(\gamma)^{-1} \qquad \forall \gamma \in G, z \in \tilde{X}.$$

*Proof.* The proof is the same as for Riemann surfaces, see for example [Flo01] Lemma 2 □

Let $X = \mathbb{C}_p^*/q^{\mathbb{Z}}$ ($|q| < 1$) be a Tate elliptic curve. A representation $\rho$ of $\pi_1^{top}(X, 0) \cong \mathbb{Z} \cong q^{\mathbb{Z}}$ is given by sending a generater to a matrix $A \in GL_r(\mathbb{C}_p)$.

Let $M_\rho$ be the vector bundle attached to $\rho$. We will show that the pullback by $N$-multiplication $N^*M_\rho$ is isomorphic to a vector bundle attached to an integral representation for some suitable integer $N$ (if we omit the corresponding connections):

**Remark 7.13.** There is an exact sequence of fundamental groups ([And03] III 2.3.2):

$$\pi_1^{temp}(\mathbb{C}_p^*) \longrightarrow \pi_1^{temp}(X,0) \twoheadrightarrow q^{\mathbb{Z}} = \pi_1^{top}(X,0)$$

For an integer $N \geq 1$ the $N$ multiplication $N : X \to X$ induces $N$-multiplication on the corresponding fundamental groups, i.e.

$$N : \pi_1^{temp}(X,0) \to \pi_1^{temp}(X,0), \qquad \gamma \mapsto N \cdot \gamma.$$

**Lemma 7.14.** *Let*

$$\rho : \pi_1^{top}(X,0) \to GL_r(\mathbb{C}_p), \qquad q \mapsto A$$

*be a discrete representation. Then the pull-back representation $N^*\rho$ induced by*

$$\pi_1^{temp}(X,0) \xrightarrow{N\cdot} \pi_1^{temp}(X,0) \twoheadrightarrow \pi_1^{top}(X,0) \xrightarrow{\rho} GL_r(\mathbb{C}_p)$$

*factors over $\pi_1^{top}(X,0)$ and is given explicitly by $q \mapsto A^N$.*

**Proposition 7.15.** *Let*

$$\rho : \pi_1^{top}(X,0) \to GL_r(\mathbb{C}_p), \qquad q \mapsto A$$

*be a $\Phi-$bounded representation corresponding to an indecomposable vector bundle. We assume that $A$ is in Jordan normal form with eigenvalues $\lambda$ on the diagonal with $0 \leq v(\lambda) < m = v(q)$ as described in Faltings theorem (see Theorem 3.50). Choose a $N \in \mathbb{N}, N > 0$ such that $N \cdot v(\lambda) = k \cdot v(q)$ for some $k \in \mathbb{N}$. Then the DeWe representation corresponding to the $N$-pullback of the $\Phi$-bounded representation $N^*\rho$ is conjugated to the pro-finite completion of the* integral *representation*

$$\rho' : \pi_1^{top}(X,0) \to GL_r(\mathfrak{o}), \qquad q \mapsto A'$$

*where $A'$ is defined by*

$$(A')^{-1} := \begin{pmatrix} \lambda^N/q^k & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda^N/q^k \end{pmatrix}$$

*Proof.* We will apply Lemma 7.12: Choose the function

$$f : \mathbb{C}_p^* \to GL_r(\mathbb{C}_p), \qquad z \mapsto z^k \cdot E_r \quad (E_r = \text{identity})$$

The matrix $A^N$ is conjugated (Jordan normal form) to the matrix

$$\tilde{A} := \begin{pmatrix} \lambda^N & q^k & 0 \\ & \ddots & q^k \\ 0 & & \lambda^N \end{pmatrix}.$$

Then we have the following relation

$$f(qz) = q^k f(z) = \underbrace{\begin{pmatrix} \lambda^N & q^k & 0 \\ & \ddots & q^k \\ 0 & & \lambda^N \end{pmatrix}}_{\tilde{A}} \cdot f(z) \cdot \underbrace{\begin{pmatrix} \lambda^N/q^k & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda^N/q^k \end{pmatrix}^{-1}}_{(A')^{-1}}.$$

Hence the vector bundles attached to the representations given by the matrices $A^N$ and $A'$ are isomorphic □

**Remark 7.16.** We assume that this relation can be extended partly to Mumford curves of genus $g \geq 2$. However it will be more difficult than in the genus one case to find suitable functions $f$ because the fundamental group is non-abelian and the universal covering is more complex than $\mathbb{C}_p^*$. If one considers line bundles one should be able to deduce results from the corresponding line bundles on the Jacobian which is $p$-adically uniformized.

**Remark 7.17.** It is true that the connections attached to *integral* $\Phi$-bounded representations coincide with connections induced by the corresponding algebraic representation. If the $\Phi$-bounded representation is *non integral*, then this is not the case as can be seen in the proof of Theorem 7.15 in the case of Tate elliptic curves. The connections attached to the representation given by the integer $\lambda^N$ is not equal to the connection corresponding to $\lambda^N/q^k$.

## 7.3 Canonical connections and a Riemann-Hilbert correspondence

$K$ a finite extension of $\mathbb{Q}_p$
$V$ its ring of integers

In this section we show that certain vector bundles in the categories $\mathfrak{B}^s_{X_{\mathbb{C}_p}}$ and $\mathfrak{B}_{A_{\mathbb{C}_p}}$ defined by Deninger and Werner are equipped with canonical connections. As an application one can combine the canonical connections with Faltings $p$-adic Simpson correspondence to obtain a $p$-adic Riemann-Hilbert correspondence.

**Theorem 7.18.** *Let $A/K$ be an abelian variety with good ordinary reduction. There is a category equivalence between the category of temperate representations of $TA$ (on $K$-vector spaces) and the category of homogeneous vector bundles on $A$. In particular, each homogeneous vector bundle has a canonical connection. The correspondence is compatible with tensor products, duals, internal homs and exterior powers.*

*If Assumption 7.9 is true, than the same is true for vector bundles and representations defined over $\mathbb{C}_p$.*

*Proof.* The equivalence of categories follows from Sections 6 and 7.1. The correspondence is compatible with tensor products, duals, internal homs and exterior powers because this is the case for the construction of Deninger-Werner (Section 3.3) □

**Theorem 7.19.** *Let $E/K$ be an elliptic curve which has either ordinary good reduction, or is a Tate elliptic curve (in the latter case we assume that the DeWe-functor is fully faithful). Then there is an equivalence of categories between the category of temperate representations of $\pi_1^{alg}(E, 0)$ and the category of homogeneous vector bundles on $E_{\mathbb{C}_p}$. In particular every homogeneous vector bundle on $E_{\mathbb{C}_p}$ admits a canonical connection.*

*Proof.* In the case of curves Assumption 7.9 is true, hence the result follows from Theorem 7.18 in the ordinary good reduction case. If $E$ is a Tate curve, then the result follows from Proposition 7.15 □

**Theorem 7.20.** *If $X/K$ is curve having good ordinary reduction and $x$ a $\overline{K}$ valued base point, then there is an equivalence of categories between the category of temperate characters (one-dimensional temperate representations) of $\pi_1(X_{\overline{\mathbb{Q}_p}}, x)$ to $\mathbb{C}_p^*$, and the category of line bundles of degree $0$ on $X_{\mathbb{C}_p}$. In particular, each line bundle of degree $0$ on $X_{\mathbb{C}_p}$ admits a canonical connection.*

*Proof.* We deduce this by pulling back line bundles on the Jacobian: Let $\mathcal{X}$ be a smooth model of $X$, and let $\mathcal{A}$ be its Jacobian. Let $j = j_P : \mathcal{X} \hookrightarrow \mathcal{A}$ be the Jacobian embedding corresponding to a point $P$. If $L$ is a line bundle of degree 0 on $X_{\mathbb{C}_p}$ then there exists a line bundle $L'$ of degree 0 on $A_{\mathbb{C}_p}$ satisfying $L = j^*_{\mathbb{C}_p} L'$. Let $\mathcal{L}'$ be a $\mathfrak{o}$-model of $L'$, then $j^*_{\mathfrak{o}}\mathcal{L}'$ is also a model of $L$. Because $\mathcal{L}'$ is associated to a temperate representation, there exists an integer $N > 0$ such that $N^*\mathcal{L}'$ can be trivialized by finite étale coverings of $\mathcal{A}$ modulo $p^n$ for every $n > 0$ ($N$ denotes the multiplication by $N$ on $\mathcal{A}$). So fix a $n > 0$, and choose an étale trivializing cover $\pi : \mathcal{A}' \to \mathcal{A}$ for $N^*\mathcal{L}'$ modulo $p^n$. Consider the following diagram of coverings:

$$\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{A},N} \mathcal{A} \times_{\mathcal{A},\pi} \mathcal{A}' & \xrightarrow[pr_2]{} & \mathcal{A}' \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
\mathcal{X} \times_{\mathcal{A},N} \mathcal{A} & \xrightarrow[pr_2]{} & \mathcal{A} \\
\downarrow \tilde{N} & & \downarrow N \\
\mathcal{X} & \xrightarrow{j} & \mathcal{A}.
\end{array}$$

Here the map $\tilde{\pi}$ is étale and the map $\tilde{N}$ is étale on the generic fiber. By the diagram there is an isomorphism

$$\tilde{\pi}^*\tilde{N}^*\mathcal{L} \cong \tilde{\pi}^*\tilde{N}^*j^*\mathcal{L}' \cong pr^*\pi^*N^*\mathcal{L}'.$$

This implies that $\tilde{\pi}^*\tilde{N}^*\mathcal{L}$ is trivial modulo $p^n$ because this is the case for $pr^*\pi^*N^*\mathcal{L}'$. It follows that $L = \mathcal{L}_{\mathbb{C}_p}$ is associated to a temperate representation of $\pi_1(X_{\overline{\mathbb{Q}_p}}, x)$ □

There is a well known relation between connections and Higgs fields on vector bundles:

**Remark 7.21.** Let $Y/S$ be a scheme smooth and of finite type over a base scheme $S$. Let $\mathcal{E}$ be a vector bundle on $Y$. Then the set of $S$-connections naturally form a $End\mathcal{E} \otimes \Omega^1_{Y/S}$ pseudotorsor (i.e. a torsor if the set of connections is not empty). If $\nabla$ and $\nabla'$ are two $S$-connections then

$$\nabla - \nabla' = \theta \in End\mathcal{E} \otimes \Omega^1_{Y/S}.$$

**Remark 7.22 (a $p$-adic Riemann-Hilbert correspondence).** We have seen that for elliptic curves with ordinary good reduction or Tate-elliptic curves each homogeneous vector bundle has a canonical connection. Also each line bundle of degree 0 on a curve with ordinary reduction has a canonical connection. In these cases one can combine Faltings' $p$-adic Simpson correspondence (Section 3.4) with Remark 7.21 and the canonical connections to obtain a $p$-adic Riemann Hilbert correspondence (if we allow $A_2(\overline{V})$-coefficients as in Faltings work).

$$(F, \nabla) \underset{\nabla - \nabla^{can} = \theta}{\longleftrightarrow} (F, (\nabla^{can} + \theta)) \longleftrightarrow (F, \theta) \underset{p\text{-adic Simpson}}{\longleftrightarrow} \rho_{(F,\theta)}$$

Here $F$ is a vector bundle with connection $\nabla$. The connection $\nabla$ can be written as $\nabla = \theta + \nabla^{can}$ for a Higgs field $\theta$ and the canonical connection $\nabla^{can}$.

It would be interesting to know how the sheaf of locally analytic function defined by Coleman and Berkovich (Section 3.10) fits into this picture if a vector bundle with connection is not attached to a topological or temperate representation.

**Example 7.23 (Horizontal sections).** If $X/\mathrm{Spec}\overline{\mathbb{Q}_p}$ is a curve then each rank one Higgs-bundle $(\mathcal{L}, \theta)$ (of degree 0) lies in the image of Faltings functor [Fal05] Section 5. By Remark 7.22 there is an equivalence of categories between the characters of $\pi_1^{alg}(X, x)$ and line bundles with connection on $X_{\mathbb{C}_p}$ (if $X$ has good ordinary reduction or is a Tate curve). The vector bundle with connection $(\mathcal{O}_X, \nabla)$ with $\nabla := d + \theta$ ($\theta$ a Higgs field) is not locally unipotent whenever $\theta \neq 0$, hence Berkovichs' $p$-adic integration does not apply directly. To find a solution $f$ (a horizontal section of $\nabla$) of the differential equation $df + \theta f = 0$ one has to exponentiate a primitive $g_\theta$ of $\theta$, i.e. we set $f = Exp(g_\theta)$. A primitive $g_\theta$ of $\theta$ that is unique up to a constant can be found using Coleman-Berkovich integration (Theorem 3.61 a), b)). One has to choose an exponential function $Exp$ because the exponential series does not converge in general (See [Rob00] Section 5.4.4). Faltings' $p$-adic Simpson correspondence depends on the choice of an exponential function, so one can take this one to make things canonical. The set $\{a \cdot Exp(g_\theta) | a \in \mathbb{C}_p\}$ is a one-dimensional $\mathbb{C}_p$-vector space of horizontal sections of the connection $\nabla = d + \theta$. The same reasoning applies to arbitrary line bundles with connection.

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