On the Monodromy of 4-dimensional Lagrangian Fibrations

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Contents

1. Introduction

1.1. **Introduction and Results.** Compact hyperkähler manifolds are a very special class of complex manifolds. This is manifest in the fact that up to deformation there are only very few examples known. As compact hyperkähler manifolds admit a holomorphic-symplectic form, they are even dimensional. In dimension two compact hyperkähler manifolds are $K3$-surfaces and in dimension four the known examples are up to deformation either Hilbert schemes of points on $K3$-surfaces or generalised Kummer varieties. Therefore one would either like to construct new examples of compact hyperkähler manifolds or else understand why there exist so few of them.

The restrictive nature of compact hyperkähler manifolds also governs the type of fibration that such manifolds admit. By a theorem, due to Matsushita [45], we know that a fibration on a compact hyperkähler manifold is a fibration in complex-Lagrangian tori. Furthermore we know that the base of such a fibration is a Fano variety with the same Hodge numbers as the complex projective space [46]. Sawon explained in [65] that and how Lagrangian fibrations could be used to classify compact hyperkähler manifolds up to deformation. In fact all of the known examples of compact hyperkähler manifolds can be deformed into compact hyperkähler manifolds that admit a Lagrangian fibration.

In this thesis we study Lagrangian fibrations on four-dimensional hyperkähler manifolds. The discriminant locus Δ of such a fibration is a curve in $\mathbb{P}^2$. Around components of Δ the fibration exhibits monodromy. Outside the discriminant locus the Lagrangian fibration endows the base $\mathbb{P}^2$ with the structure of an integral affine manifold. We are mainly interested in the discriminant locus and in the monodromy. Control over the degree of Δ might lead to a classification of Lagrangian fibrations, see [64] and [72]. For fibrations whose fibres are polarised abelian varieties, the monodromy of the Lagrangian fibration (or equivalently that of the affine structure) lies in an appropriate integral symplectic group. We study the case of principal polarisations. In this case the monodromy transformations lie in $\mathrm{Sp}(4, \mathbb{Z})$. Studying group theoretical properties of $\mathrm{Sp}(4, \mathbb{Z})$ we are able to prove a monodromy theoretical formula for the degree of the discriminant locus and an upper bound for $\deg(\Delta)$, (Theorem 3.15 and Corollary 3.25). Further we prove under additional assumptions a dichotomy for such fibrations. (Theorem 3.47).

In chapter one we first review the definition of a compact hyperkähler manifold, the known examples and some results on Lagrangian fibrations. Then we look at elliptic fibrations on $K3$-surfaces from the point of view of monodromy and finally we discuss types of unipotent monodromy transformations, in particular we focus on transformations that are “unipotent of rank one”. This means that the transformation is unipotent and that its logarithm has rank one.

In chapter two we describe the relation between the integral symplectic group $\mathrm{Sp}(4,\mathbb{Z})$ and the mapping class group Map_2 of a surface of genus two. In order to study the monodromy of principally polarised Lagrangian fibrations we study central extensions

$$0 \longrightarrow \mathbb{Z} \longrightarrow E \longrightarrow \mathrm{Sp}(4,\mathbb{Z}) \longrightarrow 1$$

of $\mathrm{Sp}(4,\mathbb{Z})$ by $\mathbb{Z}$. The latter are classified by the group cohomology group $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$. We use a presentation of Map_2 due to Wajnryb [75] and Matsumoto [49] to describe natural $\mathbb{Z}$-extensions of $\mathrm{Sp}(4,\mathbb{Z})$. This leads to two natural extensions that we denote by $\kappa^* + \gamma^*$ and γ^* that generate $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$ (Sections 2.3 and 2.4). Further we describe a central extension $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$ of $\mathrm{Sp}(4,\mathbb{Z})$ by $\mathbb{Z}^2$ and show that unipotent transformations of rank one have a distinguished lift in this central extension. By a distinguished lift of a symplectic transformation in a central extension E we mean a distinguished inverse image under the homomorphism $E \longrightarrow \mathrm{Sp}(4,\mathbb{Z})$. The distinguished lift in $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$ leads to distinguished lifts in all central $\mathbb{Z}$-extensions of $\mathrm{Sp}(4,\mathbb{Z})$.

In chapter three we study what we call monodromy factorisations of Lagrangian fibrations. Restricting a Lagrangian fibration $f : X \longrightarrow \mathbb{P}^2$ to a general line $l \subset \mathbb{P}^2$, one obtains an abelian fibration $f_{|X_l} : X_l \longrightarrow l$ with base $\mathbb{P}^1$. This fibration has singular fibres over the points where l intersects Δ. Choosing an appropriate presentation of the fundamental group $\pi_1(l \setminus \Delta)$ yields a factorisation of the identity into monodromy transformations. We call such a factorisation μ_l a monodromy factorisation of f. We make the assumption that the monodromy transformation T around a point of $l \cap \Delta$ is unipotent of rank one. This assumption puts restrictions on the singularities of the general singular fibre of $f : X \longrightarrow \mathbb{P}^2$. We show that monodromy factorisations that satisfy this assumption admit a distinguished lift in central $\mathbb{Z}$-extensions of $\mathrm{Sp}(4,\mathbb{Z})$. The distinguished lift of a monodromy factorisation is an integer and this allows us to evaluate classes in $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$ on monodromy factorisations.

Next we give geometrical interpretations of the two natural generators $\kappa^* + \gamma^*$

and γ^* of $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$. We prove that the evaluation of the class $\kappa^*+\gamma^*$ on a monodromy factorisation μ_l gives the first Chern class of the canonical extension of the Hodge bundle of $f_{|X_l}$, (Theorem 3.7). This interpretation together with Matsushitas result that $R^1 f_*\mathcal{O}_X$ is isomorphic to the cotangentbundle of the base allows us to prove that for a Lagrangian fibration with principally polarised fibres and unipotent monodromy of rank one

$$\deg(\Delta) = 30 + 2\gamma^*(\mu_l)$$

where μ_l is a monodromy factorisation of $f_{|X_l}$ (Theorem 3.15). The interpretation of the class γ^* is as follows. The fibration $f_{|X_l} : X_l \longrightarrow l$ gives rise to a moduli map

$$\varphi : l \longrightarrow \overline{\mathcal{A}}_2$$

into the compactification of the moduli space of principally polarised abelian surfaces $\mathcal{A}_2$. Assume that the general fibre of the Lagrangian fibration is not reducible as a principally polarised abelian variety. Denote by D_1 the divisor on $\overline{\mathcal{A}}_2$ that parametrises reducible principally polarised abelian varieties. Then $\deg(\varphi^* D_1)$ is given by $-\gamma^*(\mu_l)$, where $\gamma^*(\mu_l)$ denotes the evaluation of γ^* on a monodromy factorisation μ_l (Theorem 3.20). In this sense $\gamma^*(\mu_l)$ counts the fibres of $f_{|X_l} : X_l \longrightarrow l$ that are reducible as p.p.a.s..

Next we check the formula of Theorem 3.15 in an example of a Lagrangian fibration with degree of the discriminant locus equal to 30. Then we prove that if the general fibre is not reducible as a principally polarised abelian variety, then

$$\deg(\Delta) \leq 30$$

(Corollary 3.25).

Principally polarised abelian surfaces are intimately related to genus two curves. The relation is that of a principally polarised abelian surface with its theta-divisor. Using this correspondence we show that one can associate to the restriction f_{X_l} : $X_l \longrightarrow l$ of a principally polarised Lagrangian fibration to a general line l in $\mathbb{P}^2$ a genus two fibration over l (Lemma 3.18 and the discussion thereafter). Studying the genus two fibrations we prove further restrictions of the values of $\deg(\Delta)$ and γ^*. Theorem 3.27 establishes that $\gamma^*(\mu_l)$ is even and that $\deg(\Delta) \geq 10$.

The associated genus two fibration is a complex surface. We determine the numerical invariants of these surfaces and their place in the Enriques-Kodaira classification depending on the values of $\deg(\Delta)$ and γ^* (Theorem 3.33). This theorem implies that the surface is (birational to) a $K3$-surface S if and only if $\deg(\Delta) = 30$. For the case $\deg(\Delta) = 30$ Theorem 3.23 implies that the general singular fibres of the genus two fibration is irreducible with a single node. As there are no known examples of Lagrangian fibrations with principally polarised fibres and unipotent monodromy that have $\deg < 30$, it is natural to ask whether there exist such fibrations. In order to have $\deg < 30$ the associated genus two fibration must have fibres that consist in two elliptic curves that intersect in one point. Markushevich [42], [44] and Sawon [66] study fibrations in Jacobians but the genus two fibrations they consider do not contain such curves. A second question would then be if such a hyperkähler manifold is deformation equivalent to $S^{[2]}$ or K_2. A step towards the first question is the question whether the corresponding complex surface fibred by genus two curves exists. We could answer this question affirmatively by explicitly constructing an appropriate surface (Proposition 3.37 and Corollary 3.41).

In section 3.8 we try to globalise the construction of the genus two pencil. The goal is to construct a Lagrangian fibration with discriminant locus of degree 26 which might then be a new example of a compact hyperkähler fourfold. The idea is to construct a genus two fibration over $\mathbb{P}^2$ such that the relative compactified Jacobian has $\deg(\Delta) = 26$. Our construction involves the construction of a divisor B on the projectified tangent bundle $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ with certain singularities. The genus two fibration is then constructed as the double cover of $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ branched along B. We are partially successful in constructing the divisor B as required (Proposition 3.42). The divisor we construct is such that the discriminant locus of the genus two fibration has a component over which each curve has two nodes (Proposition 3.44). The problem is that contrary to the case of a family where the general singular fibre has a single node, we do not know whether the compactified Jacobian of such a family is smooth. Therefore this construction might not lead to a smooth fourfold. But we conjecture that our divisor B can be deformed to a divisor B_{new} such that the general singular fibre of the genus two fibration constructed from B_{new} has a single node. The relative compactified Jacobian would then be smooth and the discriminant locus of the abelian fibration be of degree 26. But we were unable to prove that the relative compactified Jacobian obtained in this way admits a holomorphic-symplectic form.

Next we relate our results to results of Sawon. He studies in [64] Lagrangian fibrations with polarised fibres. Under the assumption that the polarisation of the fibres comes from a global divisor on X and an assumption on the general singular fibre related to our assumption on the monodromy he proves a formula for $\deg(\Delta)$ in terms of a characteristic class of X. For X deformation equivalent to the Hilbert scheme of points on a $K3$-surface his formula yields $\deg(\Delta) = 30$. We use a recent result of Sawon [66] together with Theorem 3.23 to prove - also under the assumption that the fibrewise polarisation is induced by a global divisor - the following dichotomy for Lagrangian fibrations in principally polarised abelian surfaces (Theorem 3.47). Either the general fibre is reducible as a principally polarised abelian variety or $\deg(\Delta) = 30$ and no fibre is reducible as a principally polarised abelian variety. We conjecture that in the second case of the dichotomy provided that $f : X \longrightarrow \mathbb{P}^2$ admits a section, X is deformation equivalent to the Hilbert scheme $S^{[2]}$ of a $K3$-surface S.

1.2. **Compact hyperkähler manifolds.**

Definition 1.1. *A holomorphic 2-form σ on a complex manifold X is called holomorphic symplectic, if it is closed and everywhere non-degenerate. A complex manifold X is called holomorphic symplectic if it has a holomorphic symplectic form.*

The existence of an everywhere non-degenerate holomorphic 2-form implies that X has even dimension $\dim_{\mathbb{C}}(X) = 2n$. The non-degeneracy of σ is equivalent to the fact that σ^n is nowhere vanishing. Therefore X has trivial canonical bundle and vanishing first Chern class $c_1(X)$.

Definition 1.2. *A compact Kähler manifold X is called irreducible holomorphic symplectic, if it is simply connected and has $H^0(X, \Omega^2_X)$ generated by a everywhere non-degenerate two-form.*

On a compact Kähler manifold holomorphic p-forms are closed. Thus a two-form as in the definition is a holomorphic symplectic form. For compact Kähler manifolds with vanishing first Chern class one has the Bogomolov decomposition theorem.

Theorem 1.3. *Let X be a compact Kähler manifold with $c_1(X) = 0$. Then X has a finite unramified cover Y with*

$$Y \simeq Z \times \prod_i C_i \times \prod_j S_j,$$

where

(1) *Z is a complex torus.*
(2) *Each C_i is a Calabi-Yau manifold (i.e. simply connected, compact Kähler with $K_{C_i} = \mathcal{O}_{C_i}$ and $H^2(C_i, \mathcal{O}_{C_i}) = 0$).*
(3) *Each S_j is an irreducible holomorphic symplectic manifold.*

Proof: See [8], Exposé XVI.

□

Definition 1.4. *A compact $4n$-dimensional Riemannian manifold (M, g) is called compact hyperkähler if the holonomy group of g is $\mathrm{Sp}(n)$. The metric g is then called a hyperkähler metric.*

A compact hyperkähler manifold (M, g) admits three complex structures I, J and K such that $K = I \circ J = -J \circ I$ and g is Kähler with respect to all three of them. The corresponding Kähler forms are denoted by ω_I, ω_J and ω_K. The 2-form $\sigma := \omega_J + i\omega_K$ is a (unique up to scale) holomorphic symplectic form on the complex manifold (M, I), which is thereby irreducible holomorphic symplectic. Conversely Yau's theorem implies that each Kähler class on an irreducible holomorphic symplectic manifold contains a unique hyperkähler metric.

In this thesis we will use the terms irreducible holomorphic symplectic manifold and compact hyperkähler manifold interchangeably.

In dimension two $K3$-surfaces are the only compact hyperkähler manifolds. In higher dimensions only very few examples of compact hyperkähler manifolds are known. In each dimension $2n$ for $n \geq 2$ one has two constructions leading to compact hyperkähler manifolds, namely the Hilbert scheme of points on a $K3$-surface and the generalised Kummer varieties.

Example 1.5. *(Hilbert schemes of points on $K3$-surfaces)* Let S be a $K3$-surface. The Hilbert scheme of n points on S, $\mathrm{Hilb}^n(S)$, is a resolution of singularities of the symmetric product $S^{(n)} = S^n/\mathcal{S}_n$. In dimension two this resolution is the blow up of the diagonal in $S^{(2)}$. For a surface S the Hilbert scheme $S^{[n]} := \mathrm{Hilb}^n(S)$ is a compact complex manifold. By a result of Varouchas [74] it is Kähler provided that the surface S is. In case S is a $K3$-surface its holomorphic symplectic form induces a holomorphic symplectic form on $S^{[n]}$. One can then show that the holomorphic-symplectic form is unique up to scale and that $S^{[n]}$ is simply connected, see [5] section 6.

Example 1.6. *(Generalised Kummer varieties)* The construction of the generalised Kummer variety is similar to the foregoing example. But instead of a $K3$-surface one starts with a 2-dimensional complex torus A. As in the case of a $K3$-surface $\mathrm{Hilb}^{n+1}(A)$ is Kähler and the holomorphic symplectic form of A induces a holomorphic symplectic form on $\mathrm{Hilb}^{n+1}(A)$. But neither is $\mathrm{Hilb}^{n+1}(A)$ simply connected nor is the holomorphic symplectic form unique. The natural morphism

$$\begin{array}{ccc} A^{(n+1)} & \longrightarrow & A \\ \{p_0, ..., p_n\} & \mapsto & \sum_{i=0}^{n} p_i \end{array}$$

however induces a morphism $A^{[n+1]} \longrightarrow A$. Beauville shows in [5] that a fibre $K_n(A)$ of this morphism is smooth and simply connected and that the induced holomorphic symplectic form is unique (up to a scale). For $n = 1$ this construction gives the Kummer $K3$-surface constructed from A. For this reason the varieties K_n are called generalised Kummer varieties.

The Betti and Hodge numbers of these two standard series of examples can be calculated from the generating functions discovered by Göttsche and Soergel in [21] and [22]. The second Betti number of the Hilbert scheme is $b_2(S^{[n]}) = 23$ and that of the generalised Kummer variety is $b_2(K_n) = 7$. This implies that the two examples are not deformation equivalent. O'Grady found two examples in dimension 6 and 10 with second Betti numbers 8 and 24 respectively. Up to deformation these four examples are the only known examples of compact hyperkähler manifolds.
Therefore one would either like to construct new examples of compact hyperkähler manifolds or else understand why there are so few.

1.3. **Lagrangian fibrations and affine structures.**

The fact that a manifold X is compact hyperkähler severely restricts the fibrations that X admits. Such fibrations are described by following theorem due to Matsushita.

Theorem 1.7. *Let (X, σ) be a $2n$-dimensional compact hyperkähler manifold, σ a holomorphic-symplectic form and $f : X \longrightarrow B$ a holomorphic map with connected fibres onto a Kähler manifold B of dimension $0 < \dim B < 2n$. Then the following holds:*

1. *B is n-dimensional, projective, Fano and has the same Hodge numbers as $\mathbb{P}^n$.*
2. *Each irreducible component of a fibre is σ-Lagrangian.*
3. *Smooth fibres are n-dimensional complex tori.*

This theorem is proved in [45] and [46] for projective fibrations. Huybrechts generalised the result to the non-projective case, Proposition 24.8 in [32]. The point (3) follows from the first two as in the case of real completely integrable systems, see [26]. As it is particularly relevant to what follows, we shall nevertheless explain the proof of (3) briefly, see Proposition 1.10. It is easy to see that in dimension 4 the base of such a fibration actually equals $\mathbb{P}^2$. A connected Fano surface B has Kodaira dimension $\mathrm{kod}(B) = -\infty$. As $b_2(B) = 1$ the surface is minimal and has irregularity $q(B) = 0$, the Enriques-Kodaira classification implies that $B = \mathbb{P}^2$.

Definition 1.8. *By a Lagrangian fibration we mean a holomorphic map $f : X \longrightarrow B$ as in the above theorem. The discriminant locus of a Lagrangian fibration f is the critical locus Δ. As a set this is*

$$\Delta = \{b \in B | \exists x \in f^{-1}(b) : \mathrm{rk}\, df_x < n\}.$$

We use the following notation $X_0 = X \setminus \{x \in X | \, \mathrm{rk}\, df_x < n\}$, $B_0 = f(X_0)$, $f_0 := f_{|X_0} : X_0 \longrightarrow B_0$, $B_1 = B \setminus \Delta$, $X_1 = f^{-1}(B_1)$ and $f_1 := f_{|X_1} : X_1 \longrightarrow B_1$.

Remark 1.9. The discriminant locus Δ is a hyper-surface in B. See [33] proposition 3.1.

With the next two propositions we follow the treatment of Markushevich in [44].

Proposition 1.10. *Let $f : X \longrightarrow B$ be a Lagrangian fibration. Then the following holds.*

1. *There is a canonical isomorphism $\iota_\sigma : f_0^* \mathcal{T}^*_{B_0} \xrightarrow{\sim} \mathcal{T}_{X_0/B_0}$.*
2. *For each point $b \in B_0$ there is an action of $\mathcal{T}^*_{b,B}$ on the fibre $f_0^{-1}(b)$. For $b \in B_0$ each connected component of $f_0^{-1}(b)$ is isomorphic to a quotient of*

$\mathcal{T}^*_{B,b}$ by a lattice of rank $\leq 2n$. In particular, for $b \in B_1$ the fibre $f^{-1}(b)$ is isomorphic to a complex torus $\mathcal{T}^*_{B,b}/\Lambda^*_b$, where Λ^*_b is a lattice of rank $2n$.

(3) *Canonically associated to f is the Albanese fibration* $\mathrm{Alb}(f) : \mathrm{Alb}(X) := \mathcal{T}^*_{B_0}/\Lambda \longrightarrow B_0$*, where Λ is a well defined family of lattices in $\mathcal{T}^*_{B_0}$.*

Proof: (1) Contraction with the holomorphic symplectic form σ defines a natural isomorphism

$$\iota : \mathcal{T}^*_X \longrightarrow \mathcal{T}_X.$$

The fibration being Lagrangian and f_0 being smooth, this induces an isomorphism $f_0^*\mathcal{T}^*_{B_0} \xrightarrow{\sim} \mathcal{T}_{X_0/B_0}$.

(2) Let $\alpha_1, ..., \alpha_n$ be a local holomorphic frame in $\mathcal{T}^*_{B_0}$ in a neighbourhood U of $b \in B_0$ and $v_1, .., v_n$ the corresponding vector fields on $X_0|_U$. We can assume the α_i closed. Then the v_i commute and exponentiation gives the action of $\mathcal{T}^*_{b,B}$ on $f^{-1}(b)$. As all fibres are Lagrangian, they are all of dimension n. Let $b \in B_0$ and Z be the connected component of $z \in f_0^{-1}(b)$. As Z is n-dimensional, the orbit of z is open and closed in Z and thus equal to Z. The action is therefore transitive on Z and the isotropy group Λ^*_Z a lattice in $\mathcal{T}^*_{b,B}$. If $b \in B_1$, then $f^{-1}(b)$ is smooth and compact. Consequently the lattice Λ^*_b has rank $2n$.

(3) It suffices to show that the lattice Λ^*_Z does not depend on the connected component Z. Let Z_1 and Z_2 be two connected components of $f_0^{-1}(b)$, $b \in B_0$ and Λ^*_1, Λ^*_2 the corresponding lattices. Let $s : U \longrightarrow X_0$ be a local section of f such that $s_b \in Z_1$. For $l \in \Lambda^*_1$ we have $l(s_b) = s_b$. By the inverse function theorem l extends to a local section λ of $\mathcal{T}^*_B$, such that $\lambda(s) = s$ on a neighbourhood of b. But over a dense open subset of this neighbourhood the fibres are complex tori and so λ acts as the identity over this subset. But then it acts as the identity over the hole neighbourhood and thus $l \in \Lambda^*_2$. This implies $\Lambda^*_1 = \Lambda^*_2$.

□

Remark 1.11. In general the Albanese fibration is only locally (over B_1) isomorphic to f. But in case f has a global section the Albanese fibration is isomorphic to f over B_1, see also section 6 in [64]

Proposition 1.12. *The canonical holomorphic symplectic form on $\mathcal{T}^*_B$ descends to a holomorphic symplectic form on* $\mathrm{Alb}(X)$ *such that the Albanese fibration is Lagrangian.*

Proof: See [44], proposition 2.3.

□

Proposition 1.13. *The lattice* $\Lambda^* \subset \mathcal{T}^*_{B\setminus\Delta}$ *induces a torsion free flat connection* ∇ *on* $B \setminus \Delta$ *together with a* ∇*-parallel lattice* $\Lambda \subset \mathcal{T}_{B\setminus\Delta}$.

Proof: The local system $\Lambda^* \subset \mathcal{T}^*_{B\setminus\Delta}$ induces a flat connection ∇ in $\mathcal{T}^*_{B\setminus\Delta}$. The dual connection in $\mathcal{T}_{B\setminus\Delta}$ will also be denoted also by ∇. Let $e_1^*, ..., e_n^*$ be a local basis of Λ^*. The lattice Λ^* is a Lagrangian submanifold of $\mathcal{T}^*_{B\setminus\Delta}$ with respect to the natural holomorphic-symplectic structure, see for example [26]. Therefore the 1-forms on $B \setminus \Delta$ corresponding to the above sections are closed and thus locally they are exact, $e_i^* = dx_i$. The functions x_i form a coordinate chart and the corresponding basis of $\mathcal{T}_{B\setminus\Delta}$ consists in pairwise commuting, parallel vector fields $e_i = \frac{\partial}{\partial x_i}$. This implies that ∇ is torsion-free and the dual lattice $\Lambda \subset \mathcal{T}_{B\setminus\Delta}$ is ∇-parallel.

□

Definition 1.14. *An affine structure on a real* d*-dimensional manifold consists of an atlas whose coordinate changes are affine transformations, i.e. lie in the affine group* $GL(d, \mathbb{R}) \ltimes \mathbb{R}^d$. *We call it an integral affine structure in case the coordinate changes are in* $GL(d, \mathbb{Z}) \ltimes \mathbb{R}^d$.

Remark 1.15. The data of a torsion free flat connection ∇ is equivalent to an affine structure. A parallel lattice in the tangent bundle corresponds to an integral affine structure.

Thus

Proposition 1.16. *A Lagrangian fibration induces an integral affine structure on* $B \setminus \Delta$.

□

As we consider only integral affine structures we drop the "integral" and speak of affine structures throughout .

Let $b \in B_1$ be a non critical value of a Lagrangian fibration $f : X \longrightarrow B$. The lattice Λ_b^* is then canonically isomorphic to $H_1(f^{-1}(b), \mathbb{Z})$. After choosing a base point $b_0 \in B_1$ the smooth torus fibration $f_1 : X_1 \longrightarrow B_1$ gives rise to a monodromy representation

$$\mu : \pi_1(B_1, b_0) \longrightarrow \mathrm{SL}(H_1(f^{-1}(b_0), \mathbb{Z})).$$

We adopt the convention of writing paths like maps from right to left. The choice of a basis in $H_1(f^{-1}(b_0), \mathbb{Z})$ yields a representation

$$\mu : \pi_1(B_1, b_0) \longrightarrow \mathrm{SL}(2n, \mathbb{Z}).$$

The affine structure on B_1 also exhibits monodromy around the discriminant locus Δ. This is defined to be the composition of the coordinate changes along a path α and is thus an affine transformation. We are only interested in the linear part of this affine transformation. Therefore by affine monodromy we mean the linear part only. This is given by a representation

$$\mu_{\mathrm{aff}} : \pi_1(B_1, b_0) \longrightarrow \mathrm{SL}(2n, \mathbb{Z}).$$

These two representations are related by

$$\mu_{\mathrm{aff}} = \mu^{\;-\mathsf{T}},$$

where T denotes the transpose. We will be dealing with Lagrangian fibrations whose general fibres are polarised abelian varieties. This is to say that we assume the datum of a family of fibrewise polarisations $\omega_b \in H^{1,1}(f^{-1}(b))$ for $b \in B_1$. Such a polarisation is given by an integral symplectic form ω_b on the lattice $H_1(f^{-1}(b), \mathbb{Z})$ that is positive and of type $(1,1)$ with respect to the complex structure of $f^{-1}(b)$. This datum reduces the monodromy representation to

$$\mu : \pi_1(B_1, b_0) \longrightarrow \mathrm{Sp}(H_1(f^{-1}(b_0), \mathbb{Z}), \omega_{b_o}),$$

where $\mathrm{Sp}(H_1(f^{-1}(b_0), \mathbb{Z}), \omega_{b_o})$ is the automorphism group of the symplectic lattice

$$(H_1(f^{-1}(b_0), \mathbb{Z}), \omega_{b_o}).$$

Dually this means that B_1 has the structure of an affine symplectic manifold, i.e. the coordinate changes lie in an integral symplectic group. A polarisation is principal if there is an integral basis of the lattice such that the symplectic form ω_b has the form,

$$\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

In that case the monodromy group is $\mathrm{Sp}(2n, \mathbb{Z})$.

Definition 1.17. *A Lagrangian fibration with principally polarised fibres is a Lagrangian fibration $f : X \longrightarrow \mathbb{P}^n$ together with a family of fibrewise principal polarisations ω_b for $b \in B \setminus \Delta$.*

1.4. **Elliptic $K3$-surfaces.** If X is a $K3$-surface and $f : X \longrightarrow B$ a morphism onto a curve B, then it is easy to show that the base is $\mathbb{P}^1$ and the general fibre an elliptic curve. Namely one considers the Albanese morphism $\mathrm{Alb}(f) : \mathrm{Alb}(X) \longrightarrow \mathrm{Jac}(B)$. Then $q(X) = 0$ implies that the natural map $B \longrightarrow \mathrm{Jac}(B)$ is constant, which implies $B = \mathbb{P}^1$. The genus formula in turn implies that the general fibre has genus one. For trivial reasons the fibres are Lagrangian in this case. Lagrangian fibrations are thus higher dimensional analogues of elliptic $K3$-surfaces.

In this section we discuss elliptic $K3$-surfaces as this is the simplest case of a Lagrangian fibration. Let

$$f : X \longrightarrow \mathbb{P}^1$$

be an elliptically fibred $K3$-surface. The discriminant locus Δ consists of finitely many points $p_1, ..., p_d$. Fix a base point $b_0 \in \mathbb{P}^1 \setminus \Delta$ and let $(\alpha_1, ..., \alpha_d)$ be loops that go counterclockwise around the p_i's such that

$$\pi_1(\mathbb{P}^1 \setminus \Delta) = \langle \alpha_1, ..., \alpha_d | \prod_{i=1}^{d} \alpha_i = 1 \rangle$$

is a presentation of the fundamental group. The monodromy of the elliptic fibration lies in $\mathrm{Map}_1 = \mathrm{SL}(2, \mathbb{Z})$ and the monodromy representation

$$\mu : \pi_1(\mathbb{P}^1 \setminus \Delta) \longrightarrow \mathrm{SL}(2, \mathbb{Z})$$

is encoded in a relation

$$\prod_{i=1}^{d} \mu(\alpha_i) = 1$$

in $\mathrm{SL}(2, \mathbb{Z})$. We call such a relation a monodromy factorisation. Suppose now that the monodromy transformation around each critical value p_i is in an appropriate basis given by a matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

where $k \in \mathbb{N}$. This requirement is equivalent to the assumption that the fibration be semi-stable, see [3] p. 210. The singular fibres of f are then reduced nodal curves. More specifically the singular fibre corresponding to the above monodromy is of type I_k.

We denote the braid group on n strings by Br_n and the mapping class group of a genus one surface with b boundary components by $\mathrm{Map}_{1,[b]}$. The braid group Br_n can also be interpreted as the mapping class group of a disc with boundary and n

distinguished points. The standard generators of Br_n will be denoted $a_1, ..., a_{n-1}$, i.e. a_i is the positive half-twist that braids the $i-$ and the $(i+1)-$ string. There exists a homomorphism $\deg : \mathrm{Br}_n \longrightarrow \mathbb{Z}$ that sends each standard generator to 1. This will be called the degree homomorphism.

A surface of genus one with two boundary components as in Figure 1 is a double cover of a disc branched in four points.

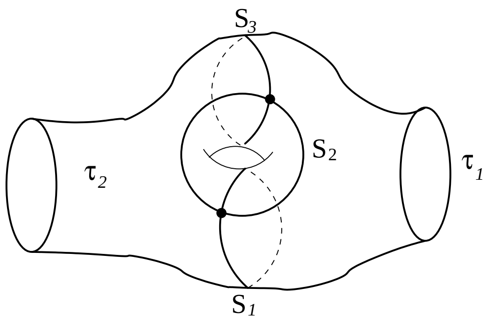

FIGURE 1. Genus one surface with two boundary components

Moving the branch points in a half-twist results in a Dehn twist upstairs, i.e. a_i corresponds to the Dehn twist along the curve S_i. This defines a surjective map $\mathrm{Br}_4 \longrightarrow \mathrm{Map}_{1,[2]}$. The braid $(a_1a_2a_3)^4$ gives a full-twist of the disc. Upstairs this results in Dehn twists along the boundary components. Thus in $\mathrm{Map}_{1,[2]}$ we have the relation

$$(a_1a_2a_3)^4 = \tau_1\tau_2, \tag{1}$$

where τ_i denotes the Dehn twist along the boundary component of the same name.

Analogously, a genus one surface with one boundary component is a double cover of a disc branched in three points and there is a surjective map $\mathrm{Br}_3 \longrightarrow \mathrm{Map}_{1,[1]}$. On the other hand, capping a boundary component of a genus two surface with two boundary components by a disc induces a map $\mathrm{Map}_{1,[2]} \longrightarrow \mathrm{Map}_{1,[1]}$. Observe that relation (1) now becomes

$$(a_1a_2)^6 = \tau_1. \tag{2}$$

This is the only relation between the generators a_1, a_2, τ_1 of $\mathrm{Map}_{1,[1]}$. Thus $\mathrm{Map}_{1,[1]} = \mathrm{Br}_3$. Glueing a disc in the remaining boundary component induces a map $\mathrm{Map}_{1,[1]} \longrightarrow \mathrm{Map}_1$ with kernel generated by the Dehn twist along the boundary component. This map now exhibits $\mathrm{Map}_1 = \mathrm{SL}(2,\mathbb{Z})$ as the quotient

$$\mathrm{SL}(2,\mathbb{Z}) = \mathrm{Br}_3 / \langle \gamma \rangle,$$

where γ is the element

$$(a_1 a_2)^6.$$

Thereby the generators a_1, a_2 map to the matrices

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with respect to the standard basis of $H_1(\Sigma,\mathbb{Z})$. We denote these matrices also by a_1, a_2. Now the element γ is central in Br_3 and thus

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Br}_3 \longrightarrow \mathrm{SL}(2,\mathbb{Z}) \longrightarrow 1$$

is a central extension. On the other hand we have a natural central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{SL}}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{Z}) \longrightarrow 1$$

of $\mathrm{SL}(2,\mathbb{Z})$ given by the pullback of the universal covering

$$0 \longrightarrow \pi_1(\mathrm{SL}(2,\mathbb{R})) \longrightarrow \widetilde{\mathrm{SL}}(2,\mathbb{R}) \longrightarrow \mathrm{SL}(2,\mathbb{R}) \longrightarrow 1$$

under the inclusion $\mathrm{SL}(2,\mathbb{Z}) \hookrightarrow \mathrm{SL}(2,\mathbb{R})$. One can show that this extension is isomorphic to the one above. We want to use this extension to study the monodromy of a semi-stable elliptic $K3$-surface. The monodromy factorisation is then

$$\prod_{i=1}^{d} \mu(\alpha_i) = 1$$

where $\mu(\alpha_i) = t_i^{n_i}$ with t_i conjugate to

$$a_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{Z})$. For a transformation of this latter type exists a distinguished lift in $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$, where by a lift we simply mean an inverse image under the natural map. Namely let $t_i = b * a_2$, where the star stands for conjugation. Then its distinguished lift $\widetilde{t_i}$ is given by $\beta * a_2$, where β is an arbitrary lift of b in $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$. As γ is central this is well defined. These lifts have

$$\deg(\tilde{t_i}) = 1.$$

Note that the degree homomorphism restricted to the kernel $\mathbb{Z}\langle\gamma\rangle$ is given by $\mathbb{Z} \xrightarrow{12} \mathbb{Z}$. We define the distinguished lift of a monodromy factorisation $\mu : \prod_{i=1}^{d} t_i^{n_i} = 1$ to be the product of the distinguished lifts of its factors

$$\prod_{i=1}^{d} \widetilde{t_i^{n_i}}.$$

This is an element of the kernel and thus an integer. The corresponding number will be denoted by

$$\gamma^*(\mu) = \gamma^*(\prod_{i=1}^{d} t_i^{n_i}).$$

Proposition 1.18. *Let $f : X \longrightarrow \mathbb{P}^1$ be a semi-stable elliptic $K3$-surface. Then*

$$\gamma^*(\prod_{i=1}^{d} t_i^{n_i}) = \chi_{\text{top}}(S^2),$$

where $\mu : \prod_{i=1}^{d} t_i^{n_i} = 1$ is a monodromy factorisation and $\chi_{\text{top}}(S^2)$ denotes the Euler number of S^2.

Originally this is a result of Moishezon and Livné [52]. Alternatively see [39] p.27f or [40] for a proof in terms of affine structures on closed surfaces.

□

This implies

$$\begin{aligned} \chi_{\text{top}}(S^2) &= \gamma^*(\prod_{i=1}^{d} t_i^{n_i}) \\ &= \frac{1}{12} \deg \left(\prod_{i=1}^{d} \widetilde{t_i^{n_i}} \right) \\ &= \frac{1}{12} \sum_{i=1}^{d} n_i. \end{aligned}$$

From which we conclude the following

Corollary 1.19.

$$\deg \Delta = \sum_{i=1}^{d} n_i = 24$$

□

Of course we knew this in advance since by the Noether formula $\chi_{\text{top}}(X) = 24$ and

$$\begin{aligned}\chi_{\text{top}}(X) &= \chi_{\text{top}}(S^2 \setminus \Delta)\chi_{\text{top}}(F_{\text{sm}}) + \sum_{i=1}^{d} \chi_{\text{top}}(F_{\text{sing}}) \\ &= \sum_{i=1}^{d} \chi_{\text{top}}(F_{\text{sing}}) \\ &= \sum_{i=1}^{d} n_i.\end{aligned}$$

Nevertheless the above reasoning will serve as a toy model for the study of Lagrangian fibrations in dimension 4. We remark that Kontsevich and Soibelmann use this kind of reasoning to prove that there are no affine structures with this type of monodromy on closed surfaces other than the sphere or the torus, see [39].

1.5. **Monodromy.** Let $f : X \longrightarrow B$ be a Lagrangian fibration. Lagrangian fibrations are equidimensional and therefore flat. For a curve $C \subset B$ the restriction $f_{|X_C} : X_C \longrightarrow C$ is thus a degeneration of complex tori. Let $C_1 \subset C$ be the part over which the fibration is smooth. The variation of Hodge structure associated to the smooth family $f_1 := f_{|X_{C_1}} : X_{C_1} \longrightarrow C_1$ has monodromy around the points of $C \setminus C_1$. By the monodromy theorem the monodromy transformations T are quasi-unipotent, see [27], p. 41 and [67]. This means that there exist $l, m \in \mathbb{N}$ such that

$$(T^l - I)^m = 0.$$

The transformation T is called unipotent in case l can be chosen to be one. In that case the smallest number $m \in \mathbb{N}$ such that $(T - I)^{m+1} = 0$ is the index of unipotency of T. For a unipotent T, one can define its logarithm by

$$N := \log T = (T - I) - (T - I)^2/2 + \ldots + (-1)^{m+1}(T - I)^m/m.$$

In general the monodromy T has a Jordan decomposition

$$T = T_{ss} \cdot T_u,$$

where T_u is unipotent and T_{ss} is semi-simple. The semi-simple part T_{ss} is of finite order. Furthermore the monodromy theorem implies that the index of unipotency does not exceed the weight of the Hodge structure under consideration. In case of an abelian fibration, the relevant Hodge structure has weight one. So if the monodromy is non-trivial, the index of unipotency has got to be one. For a Lagrangian fibration $f : X \longrightarrow \mathbb{P}^2$ with principally polarised fibres, we get a polarised variation of Hodge structure, see Section 3.2, and the monodromy transformations lie in the symplectic group, $\mathrm{Sp}(4, \mathbb{Z})$. Let $T \in \mathrm{Sp}(4, \mathbb{Z})$ be a unipotent and non-trivial monodromy transformation. Its logarithm is then $N = T - I$. As $N^2 = 0$, the rank $\operatorname{rk} N$ is either one or two.

Definition 1.20. $T \in \mathrm{Sp}(4, \mathbb{Z})$ *is unipotent of rank one if* T *is unipotent with index of unipotency one and* $\operatorname{rk}(T - I) = 1$.

We use the following convention of the symplectic form. A basis (e_1, f_1, e_2, f_2) of $\mathbb{Z}^4$ is called symplectic with respect to ω if the symplectic form in this basis is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Proposition 1.21. *Let* $T \in \mathrm{Sp}(4, \mathbb{Z})$ *be unipotent of rank one. Then* T *is conjugate in* $\mathrm{Sp}(4, \mathbb{Z})$ *to a matrix*

$$\begin{pmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $k \in \mathbb{Z}$.

Proof: Let $N = T - I$. If we choose a generator v of $\operatorname{im} N$, N becomes a map from $\mathbb{Z}^4$ to $\mathbb{Z}$

$$N : \mathbb{Z}^4 \longrightarrow \mathbb{Z}\langle v \rangle$$

We can write this as

$$Nx = \omega(u, x)v,$$

where ω is the symplectic form and u a primitive element of $\mathbb{Z}^4$. Now T has the form

$$x \mapsto x + \omega(u, x)v$$

and as T is symplectic, we get for all $x, y \in \mathbb{Z}^4$

$$\begin{aligned} \omega(x, y) &= \omega(Tx, Ty) \\ &= \omega(x, y) + \omega(x, v)\omega(u, y) + \omega(u, x)\omega(v, y) \end{aligned}$$

which implies that the bilinear form $\omega(x,v)\omega(u,y)$ is symmetric. This is equivalent to the matrix $u \cdot v^{\mathsf{T}}$ being symmetric, which in turn yields $av = bu$ for non-zero integers a and b. As u is primitive, we know that

$$\mathbb{Z}u = \mathbb{Q}u \cap \mathbb{Z}^4$$

From this we conclude that $\frac{b}{a}$ is an integer k. So far we proved that T has the form

$$x \mapsto x + k\omega(u,x)u$$

with u unique up to sign. As $v = ku$ lies in $\operatorname{im} N$, there must be a $w \in \mathbb{Z}^4$ such that $\omega(u,w) = 1$. On the subspace $U := \operatorname{span}_{\mathbb{Z}}(u,w)$ T is given by

$$\begin{aligned} u &\overset{T}{\mapsto} u \\ w &\overset{T}{\mapsto} w + ku \end{aligned}$$

Because this subspace is unimodular (i.e.

$$\omega|_U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to the given basis), $U^{\perp}$ will likewise be unimodular and $\mathbb{Z}^4 = U \oplus U^{\perp}$. Let (v_1, v_2) be a basis of $U^{\perp}$ such that $\omega(v_1, v_2) = 1$. On this subspace T is given by

$$\begin{aligned} v_1 &\overset{T}{\mapsto} v_1 \\ v_2 &\overset{T}{\mapsto} v_2 \end{aligned}$$

as $u \perp U^{\perp}$. Thus (u, w, v_1, v_2) is a symplectic basis in $\mathbb{Z}^4$ with respect to which T has the required form.

$\square$

Definition 1.22. *A transformation in* $\mathrm{Sp}(4,\mathbb{Z})$ *of the form*

$$x \mapsto x + k\omega(v,x)v$$

for a vector $v \in \mathbb{Z}^4$ *and* $k \in \mathbb{Z}$ *is called a symplectic transvection. In case* $k = 1$ *we denote the symplectic transvection by* t_v*. A symplectic transvection is called simple if* v *is primitive and* $k = 1$.

As we only deal with symplectic transvections we drop the word "symplectic" and speak of transvections throughout.

Let $T \in \mathrm{Sp}(4,\mathbb{Z})$ be a unipotent monodromy transformation and $N = T - I$ its logarithm. By the monodromy theorem $N^2 = 0$. So $\operatorname{im} N \subset \ker N$ and thus $\operatorname{rk} N \leq 2$. As N is the logarithm of $T \in \mathrm{Sp}(4,\mathbb{Z})$ it lies in the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ and thus satisfies $\omega(Nx,y) = -\omega(x,Ny)$. It follows that $\operatorname{im} N$ is isotropic. Therefore there exists a basis (e_1,e_2,f_1,f_2) of $(\mathbb{Z}^4,\omega)$ such that in this basis

$$\omega = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix},$$

for a symmetric matrix $S = \begin{pmatrix} k & m \\ m & l \end{pmatrix}$. As we assume T to be a monodromy transformation of a degeneration, Hodge theory implies that S is positive semi-definite, see [28], Proposition 13.3. Changing the basis to (e_1,f_1,e_2,f_2):

$$\omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & k & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & m & 0 & l \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $\operatorname{rk} N = 1$, T is a transvection and the positive semi-definiteness of S implies (without loss of generality) that k is positive.

Lemma 1.23. *If a transvection $t \in \mathrm{Sp}(4,\mathbb{Z})$ is a monodromy transformation of a degeneration, then k is non-negative.*

□

Let $\operatorname{rk} N = 2$. Then the positive semi-definiteness of S implies positive definiteness. Thus $k, l > 0$ and $kl > m^2$. Suppose $m = 0$. Then k and l are positive and T can be written as a product of two commuting transvections. Namely

$$T = t_{f_1}^k \cdot t_{f_2}^l.$$

That t_{f_1} and t_{f_2} commute, follows from $\omega(f_1, f_2) = 0$.

Lemma 1.24. *Suppose $T \in \mathrm{Sp}(4, \mathbb{Z})$ can be written as a product of two commuting transvections $T = t_v^k \cdot t_{v'}^l$ with $k, l > 0$. Then v, v' are uniquely determined (up to sign) and k and l are uniquely determined.*

Proof: It is easy to see that two simple transvections $t_v, t_{v'}$ commute if and only if $\omega(v, v') = 0$. Let $t = t_v, t' = t_{v'}$ and $t_1 = t_{v_1}, t_1' = t_{v_1'}$ be two pairs of commuting simple transvections such that

$$t^k t'^j = t_1^{k_1} t_1'^{j_1} =: T,$$

with $k, j, k_1, j_1 > 0$.

$$t^k t'^j(x) = x + k\omega(v, x)v + j\omega(v', x)v'$$

and thus

$$k\omega(v, x)v + j\omega(v', x)v' = k_1\omega(v_1, x)v_1 + j_1\omega(v_1', x)v_1' \tag{3}$$

for all $x \in \mathbb{Z}^4$. As v, v' are both primitive, $\mathrm{span}(v, v')$ is primitive and isotropic. Thus we can extend (v, v') to a symplectic basis (v, w, v', w') of $\mathbb{Z}^4$. And analogously there exists a symplectic basis (v_1, w_1, v_1', w_1') of $\mathbb{Z}^4$. Let $N := T - I$. Then $imN = \mathrm{span}(kv, jv') = \mathrm{span}(k_1v_1, j_1v_1')$. As imN is isotropic, we see that

$$\omega(v, v_1) = \omega(v, v_1') = \omega(v', v_1) = \omega(v', v_1') = 0,$$

from which we deduce that

$$\mathrm{span}(v, v') = \mathrm{span}(v_1, v_1').$$

Therefore

$$v_1 = av + cv' \tag{4}$$

$$v_1' = bv + dv' \tag{5}$$

for $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Inserting w_1 and w_1' in equation (3) yields

$$\begin{aligned} k_1v_1 &= k\omega(v, w_1)v + j\omega(v', w_1)v' \\ j_1v_1' &= k\omega(v, w_1')v + j\omega(v', w_1')v'. \end{aligned}$$

From equation (4) and (5) we get

$$\begin{aligned}
\omega(v,w_1) &= d\\
\omega(v,w_1') &= -c\\
\omega(v',w_1) &= -b\\
\omega(v',w_1') &= a.
\end{aligned}$$

So

$$\begin{aligned}
k_1a &= kd\\
k_1c &= -jb\\
j_1b &= -kc\\
j_1d &= ja.
\end{aligned}$$

Let k be minimal among the k,j,k_1,j_1. As $ad-bc=1$

$$k = k(ad-bc) = k_1a^2 + j_1b^2.$$

As $j_1b^2 \geq 0$, $k \geq k_1a^2$. There are two cases: $a=0$ and $a=\pm1$. Suppose $a=0$. Then $d=0$ and $A = \begin{pmatrix} 0 & \pm1 \\ \mp1 & 0 \end{pmatrix}$. In the second case $k_1 = k$. Which in turn implies $d=a=\pm1$ and $A = \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}$. So, in the second case: $v_1 = \pm v$, $v_1' = \pm v'$ and $k_1 = k$, $j_1 = j$. And in the first case: $v_1 = \pm v'$, $v_1' = \mp v$ and $k_1 = j$, $j_1 = k$.

□

2. Group theory

We studied the monodromy of elliptic $K3$ surfaces using a central extension of the monodromy group $\mathrm{SL}(2,\mathbb{Z})$. Ultimately we want to do the same in the case of 4-dimensional Lagrangian fibrations. For this we need a combinatorial description of the group $\mathrm{Sp}(4,\mathbb{Z})$.

2.1. Map_g **and** $\mathrm{Sp}(2g,\mathbb{Z})$**.** The group $\mathrm{Sp}(2g,\mathbb{Z})$ is closely connected with the mapping class group of a smooth surface of genus g (in this part what we understand by surface will be a real surface). Let $\Sigma_{g,[b],n}$ denote a compact, oriented surface of genus g with b boundary components and n distinguished points. The mapping class group $\mathrm{Map}_{g,[b],n}$ of $\Sigma_{g,[b],n}$ is then the group of isotopy classes of orientation preserving diffeomorphisms that fix the n-points and restrict to the identity on the boundary components,

$$\mathrm{Map}_{g,[b],n} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,[b],n}))$$

In case $b = 0$ or $n = 0$ we delete the corresponding indices. The mapping class group of a surface Σ acts on the homology Σ. This induces a homomorphism

$$\xi : \mathrm{Map}(\Sigma) \longrightarrow \mathrm{Sp}(H_1(\Sigma,\mathbb{Z}),\omega)$$

where ω denotes the intersection pairing. It is well known that this is an epimorphism, see [41]. Let C be a simple closed curve on Σ and N a neighbourhood of C that is homeomorphic to a cylinder. Assume N to be parametrised by cylindrical coordinates $(y,\theta) \in [-1,1] \times [0,2\pi)$, where the y-axis is the axis of the cylinder, and such that C is $(y = 0)$. The map of the cylinder onto itself given by

$$(y,\theta) \longrightarrow (y, \theta + \pi(y+1))$$

extended by the identity outside of N defines a diffeomorphism of Σ. The resulting map τ_C is called the Dehn twist along C. Isotopy classes of Dehn twists generate the mapping class group.

Theorem 2.1 (Dehn [18])**.** *The mapping class group* $\mathrm{Map}_{g,[b],n}$ *is generated by isotopy classes of Dehn twists along simple closed curves.*

Remark 2.2. Let $\tau \in \mathrm{Map}_{g,[b],n}$ be the class of a Dehn twist along a simple closed curve C, then τ acts on the homology as the symplectic transvection $t_v : x \mapsto x + \omega(v,x)v$, where $v \in H_1(\Sigma,\mathbb{Z})$ is the homology class represented by C.

Let $\Sigma_{g,[1]}$ be a surface with one boundary component $\partial\Sigma_{g,[1]}$. Let $\Sigma_{g,1}$ be the surface obtained by gluing a disc with a distinguished point into $\partial\Sigma_{g,[1]}$. The Dehn twist $\tau_{\partial\Sigma_{g,[1]}}$ around the boundary is a central element in $\mathrm{Map}_{g,[1]}$.

Theorem 2.3. *The map $f_1 : \mathrm{Map}_{g,[1]} \longrightarrow \mathrm{Map}_{g,1}$ defined by capping $\partial\Sigma_{g,[1]}$ with a disc and extending each map over the disc by the identity is an epimorphism with kernel $ker f_1 \simeq \mathbb{Z}$ generated by $\tau_{\partial\Sigma_{g,[1]}}$.*

For a proof see [75], p. 172. Let $\Sigma := \Sigma_g$ be a closed oriented surface with base point z_0. There is a natural homomorphism

$$\Psi : \mathrm{Aut}(\pi_1(\Sigma)) \longrightarrow \mathrm{Aut}(H_2(\pi_1(\Sigma), \mathbb{Z}))$$

As the surface Σ is an Eilenberg-MacLane space for its fundamental group, we know that there is a natural isomorphism

$$H_2(\pi_1(\Sigma), \mathbb{Z}) = H_2(\Sigma, \mathbb{Z}).$$

Thus $H_2(\Sigma, \mathbb{Z}) = \mathbb{Z}$. Therefore the kernel of Ψ is a subgroup of index 2 in $\mathrm{Aut}(H_2(\pi_1(\Sigma), \mathbb{Z}))$. We will denote it by $\mathrm{Aut}^+(\pi_1(\Sigma))$. Let $\mathrm{Inn}(\pi_1(\Sigma))$ be the group of inner automorphisms of $\pi_1(\Sigma)$ and $\mathrm{Out}^+(\pi_1(\Sigma))$ the quotient of $\mathrm{Aut}^+(\pi_1(\Sigma))$ by $\mathrm{Inn}(\pi_1(\Sigma))$.

Theorem 2.4. *For $g \geq 2$ there is an exact sequence*

$$1 \longrightarrow \pi_1(\Sigma) \longrightarrow \mathrm{Map}_{g,1} \longrightarrow \mathrm{Map}_g \longrightarrow 1. \tag{6}$$

The group $\mathrm{Map}_{g,1}$ can be identified with $\mathrm{Aut}^+(\pi_1(\Sigma))$. The map $\pi_1(\Sigma) \longrightarrow \mathrm{Map}_{g,1}$ maps $\alpha \in \pi_1(\Sigma)$ to the inner automorphism $i_\alpha : \pi_1(\Sigma) \ni \gamma \mapsto \alpha\gamma\alpha^{-1} \in \pi_1(\Sigma)$. Under this identification the above sequence becomes the natural exact sequence

$$1 \longrightarrow \mathrm{Inn}(\pi_1(\Sigma)) \longrightarrow \mathrm{Aut}^+(\pi_1(\Sigma)) \longrightarrow \mathrm{Out}^+(\pi_1(\Sigma)) \longrightarrow 1.$$

This theorem is found in [68], p.11. The exact sequence (6) is known as the Birman Exact Sequence, see [11], Theorem 4.3. The evaluation map

$$\begin{array}{rcl} ev_{z_0} : \mathrm{Diff}^+(\Sigma) & \longrightarrow & \Sigma \\ \varphi & \mapsto & \varphi(z_0) \end{array}$$

endows $\mathrm{Diff}^+(\Sigma)$ with the structure of a principal fibre bundle with structure group $\mathrm{Diff}^+(\Sigma, z_0)$. This gives the long exact sequence of homotopy groups:

$$\ldots \longrightarrow \pi_1(\Sigma) \longrightarrow \pi_0(\mathrm{Diff}^+(\Sigma), z_0) \longrightarrow \pi_0(\mathrm{Diff}^+(\Sigma)) \longrightarrow 1.$$

The connected component of the identity $\mathrm{Diff}_0^+(\Sigma)$ is contractible for $g \geq 2$. The above sequence therefore yields the exact sequence (6). The second assertion is the Dehn-Nielsen-Baer theorem, see for example [16] p. 84, which states that the natural homomorphism that associates to a diffeomorphism $\phi \in \mathrm{Diff}^+(\Sigma, z_0)$ the automorphism $\phi_* \in \mathrm{Aut}^+(\pi_1(\Sigma))$ induces an isomorphism $\mathrm{Map}_{g,1} \longrightarrow \mathrm{Aut}^+(\pi_1(\Sigma))$.

Remark 2.5. The map $i : \pi_1(\Sigma) \longrightarrow \mathrm{Map}_{g,1}$ is the following. Let α be a loop representing an element of $\pi_1(\Sigma)$. Then there exists an isotopy ϕ_t in $\mathrm{Diff}^+(\Sigma)$ such that $\phi_t(z_0) = \alpha(t)$ for $t \in [0,1]$. The class of ϕ_1 in $\mathrm{Map}_{g,1}$ is $i(\alpha)$. Consider the special case that α is represented by a smooth embedded curve. Let N be a cylindrical neighbourhood of α and denote the two boundary components of N by α^+ and α^-. Then the Dehn twists $\tau_{\alpha^\pm}$ are well defined in $\mathrm{Map}_{g,1}$ and

$$i(\alpha) = \tau_{\alpha^+} \circ \tau_{\alpha^-}^{-1}.$$

According to Humphreys [30] the mapping class groups $\mathrm{Map}_{g,[1]}$ and Map_g are generated by the $2g+1$ Dehn twists around the simple closed curves S_i from Figure 2.

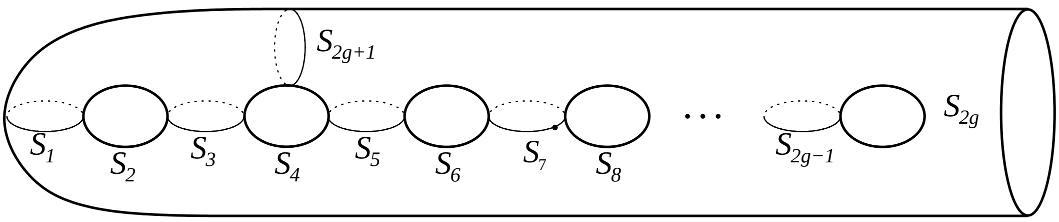

FIGURE 2. Humphries generators

Wajnryb gives in [75] a presentation of the mapping class groups $\mathrm{Map}_{g,[1]}$ and Map_g using Humpries generators. Wajnrybs presentation of $\mathrm{Map}_{g,[1]}$ contains three kinds of relations. Because of their pictorial description these relations are called:

(B) braid relations
(K) chain relation, and
(L) lantern relation.

Wajnryb shows that from this presentation one gets a presentation of Map_g, by adding the so called

(H) hyperelliptic relation.

Wajnrybs presentation was simplified by Matsumoto using Artin braid groups. We briefly describe this, as we are going to use Matsumotos presentation. Let Γ be a graph with a finite set of vertices I. Assume that Γ has no loops and that any two vertices are connected by at most finitely many edges.

Definition 2.6. *The Artin braid group associated with* Γ *is the group* $\mathrm{Br}(\Gamma)$ *generated by elements* $\{a_i | i \in I\}$*, so that if* $i, j \in J$ *are two distinct vertices connected by* k_{ij} *edges, then* a_i *and* a_j *satisfy the relation*

$$a_i a_j a_i a_j \cdots = a_j a_i a_j a_i \ldots$$

where both sides are words of length $k_{ij} + 2$*. The Coxeter-Weyl group* $W(\Gamma)$ *of* Γ *is obtained by adding the relation* $a_i^2 = 1$ *for each* i*. The length of an element* $w \in W(\Gamma)$ *is the minimal length of a word for* w*. The Coxeter number of* $W(\Gamma)$ *is the smallest* $h \in \mathbb{N}$ *such that* $(\prod_{i \in I} a_i)^h = 1$ *in* $W(\Gamma)$*.*

In case Γ is connected and $W(\Gamma)$ finite, there exists a unique longest element w_0 in $W(\Gamma)$. The Dynkin diagram A_{n-1} gives the braid group Br_n on n strings, with the a_i interpreted as the standard generators.

Remark 2.7.

i) Note that by specifying a graph Γ one specifies also a presentation of $\mathrm{Br}(\Gamma)$. The pair $(\mathrm{Br}(\Gamma), \{a_i\}_{i \in I})$ of an Artin braid group together with a set of generators is called an Artin system.

ii) Given an Artin system one obtains a natural homomorphism

$$\deg : \mathrm{Br}(\Gamma) \longrightarrow \mathbb{Z}$$

which send each generator to 1 and which is called the degree homomorphism.

iii) If $\Lambda \subset \Gamma$ is a full subgraph, then there is a natural inclusion $\mathrm{Br}(\Lambda) \subset \mathrm{Br}(\Gamma)$ [49].

Theorem 2.8 (Brieskorn-Saito, Deligne)**.** *Let* Γ *be a Dynkin diagram. Consider the following properties of an element* $w \in \mathrm{Br}(\Gamma)$*:*

- *There is a positive word in the* a_i*'s for* w*.*
- *There is a positive word in the* a_i*'s for* $a_i^{-1} w$*, for any* a_i*.*

Then there exists a unique element $\Delta(\Gamma)$ *satisfying these properties, which is minimal in the sense that if* w *satisfies the above properties, then* $\Delta(\Gamma)^{-1}w$ *has a positive word presentation in the* a_i*'s. It has the following properties:*

i) $\Delta(\Gamma)$ *is mapped to the longest element* w_0 *in* $W(\Gamma)$ *and its degree equals the length of* w_0.

ii) *The center of* $\mathrm{Br}(\Gamma)$ *is isomorphic to* $\mathbb{Z}$ *and generated either by* $c(\Gamma) = \Delta(\Gamma)^2 = \prod^h$ *or by* $c(\Gamma) = \Delta(\Gamma) = \prod^{\frac{h}{2}}$*, where* h *is the Coxeter number and* $\prod$ *is a product of all* a_i*'s with an arbitrary order.*

For a proof see [12] section 7 or [19]. By the corollary in [12] section 7.2, for the braid group $\mathrm{Br}_n = \mathrm{Br}(A_{n-1})$

$$c(A_{n-1}) = \Delta(A_{n-1})^2 = (a_1 \cdot \ldots \cdot a_{n-1})^n$$

This element corresponds to a full-twist of the disc around all the n points. Denote the following graph by T_g.

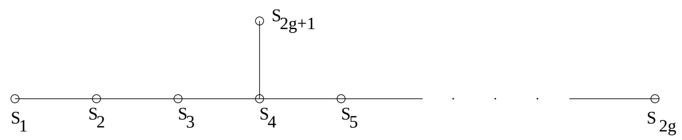

FIGURE 3. T_g

The Wajnryb-Matsumoto presentation of the mapping class group stems from this graph and its subgraphs.

Theorem 2.9 (Wajnryb-Matsumoto)**.** *A presentation of the mapping class group* $\mathrm{Map}_{g,[1]}$ *is given by*

$$\mathrm{Map}_{g,[1]} = \mathrm{Br}(T_g)/\langle \kappa, \lambda \rangle$$

where relation κ *and* λ *only apply in case* $g \geq 2$ *and* $g \geq 3$ *respectively. The relations are*

(κ) $c(A_5) = c(A_4)^2$

(λ) $c(E_7) = c(E_6)$

The generators a_i *can be interpreted as Dehn twists along the curves* S_i *in Figure 2. In case* $g = 1$ *the kernel of* $\mathrm{Map}_{g,[1]} \longrightarrow \mathrm{Map}_g$ *is the central free abelian group generated by* $(a_1a_2)^6$*. In case* $g = 2$ *this kernel is normally generated by the commutator* $[a_5, \Delta(A_4)^2]$.

For a proof see [49].

Remark 2.10.

i) Note that $\mathrm{Map}_{1,[1]} = \mathrm{Br}_3$ and that the kernel of $\mathrm{Map}_{1,[1]} \longrightarrow \mathrm{Map}_1$ is generated by $(a_1a_2)^6$. The latter element corresponds to the Dehn twist along the boundary.

ii) In case $g \geq 2$ the element $\Delta(A_{2g+1})^2$ corresponds to the hyperelliptic involution. Thus for $g = 2$ it commutes in Map_2 with a_5.

iii) In case $g = 2$ the Artin braid group $\mathrm{Br}(T_2)$ coincides with Br_6. The relation (κ) is explicitly

$$(a_1a_2a_3a_4)^{10}(a_1a_2a_3a_4a_5)^{-6} = 1$$

whereas hyperelliptic relation (π) is

$$[(a_1a_2a_3a_4)^5, a_5] = 1.$$

2.1.1. *The Torelli group.* For $g \geq 2$ the Torelli group $\mathcal{J}_g$ is known as the group that consists of those mapping classes that act trivially on the homology, i.e.

$$1 \longrightarrow \mathcal{J}_g \longrightarrow \mathrm{Map}_g \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1$$

An embedded curve σ in Σ such that the complement of σ is not connected is called a separating curve. See Figure 4 for a separating curve in case $g = 2$.

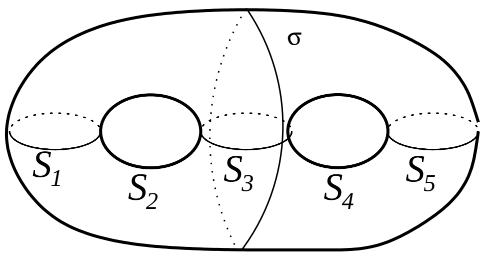

FIGURE 4. Separating curve

Such curves are boundaries and therefore a Dehn twist along them acts trivially on the homology.

Theorem 2.11. *The Torelli group for $g = 2$ is a free group and it is generated by the isotopy classes of Dehn twists along separating curves.*

A proof can be found in [51]. In case $g = 2$ Dehn twists along separating curves are pairwise conjugated. $\mathcal{J}_2$ is thus normally generated by the class of a Dehn twist along a single separating curve σ. Therefore one gets a presentation of $\mathrm{Sp}(4,\mathbb{Z})$ by adding one relation to the Wajnryb-Matsumoto presentation of Map_2. We call this relation the Torelli relation.

Lemma 2.12. *$\mathcal{J}_2$ is normally generated in Map_2 by the element*

$$\gamma = (a_1 a_2)^6.$$

Proof: A smooth genus 2 surface Σ is a double cover of the sphere S^2 branched in 6 points. Let σ be a curve in Σ that separates Σ in two surfaces Σ_1, Σ_2 each of genus one with one boundary component. Each surface Σ_i is the double cover of a disc $D_i \subset S^2$ branched in 3 points. Interpret Br_6 as the mapping class group of a disc with 6 distinguished points and the first three points as the branch points that lie in D_1. Now a_1, a_2 generate $\mathrm{Br}_3 \subset \mathrm{Br}_6$ and the word $\Delta^4(\mathrm{Br}_3) = (a_1 a_2)^6$ corresponds to the Dehn twist along the boundary of Σ_1.

□

There is a unique homomorphism from $\mathcal{J}_2$ to $\mathbb{Z}$ that sends γ to 1 and is invariant under conjugation with elements of Map_2.

Definition 2.13. *We denote this homomorphism by*

$$\deg_\gamma : \mathcal{J}_2 \longrightarrow \mathbb{Z}$$

and call it γ-degree.

The form in which we will use the Wajnryb-Matsumoto-presentation is the following. We have seen the sequence of epimorphisms

$$\mathrm{Br}_6 \longrightarrow \mathrm{Map}_{2,[1]} \xrightarrow{f} \mathrm{Map}_2 \xrightarrow{\xi} \mathrm{Sp}(4,\mathbb{Z}) \tag{7}$$

The images of the standard generators $a_1, ..., a_5$ of Br_6 in these quotients will also be denoted by a_i. In case there is risk of confusion we write for example $a_i^{\mathrm{Map}_2}$ to indicate the group we are in. The kernels of the above homomorphisms are normally generated by the following elements:

(1) $\kappa = (a_1 a_2 a_3 a_4)^{10} (a_1 a_2 a_3 a_4 a_5)^{-6}$
(2) $\pi = [(a_1 a_2 a_3 a_4)^5, a_5]$
(3) $\gamma = (a_1 a_2)^6$

We denote by N the kernel of the natural map $\mathrm{Br}_6 \longrightarrow \mathrm{Sp}(4,\mathbb{Z})$. N is normally generated by the elements κ, π and γ

$$N = \langle \kappa, \pi, \gamma \rangle. \tag{8}$$

2.2. Central extensions of $\mathrm{Sp}(4, \mathbb{Z})$.

2.2.1. *Group extensions and Cohomology.* An (abelian) extension of a group G by a group A is a short exact sequence of groups

$$0 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1,$$

such that A is abelian. Given such an extension G acts on A by conjugation, i.e. $n^g = \alpha(g)^{-1} n \alpha(g)$ for an arbitrary section $\alpha : G \longrightarrow E$. The extension is called central if A is contained in the center of E. This is the case if and only if A is the trivial G-module. Conversely one may fix a G-module structure on A and study the extensions that induce this structure. Associated to a G-module A are cohomology groups $H^i(G, A)$. $H^0(G, A) = A^G$ is the group of invariants. The semi-direct product $G \ltimes A$, given by $G \times A$ with the multiplication

$$(g, n) \cdot (h, m) = (gh, n^h + m)$$

is the unique (up to isomorphism) extension that splits. $H^1(G, A)$ classifies the splittings of $G \ltimes A$. In case the extension is central, this is just $\mathrm{Hom}(G, A)$. Extensions themselves are classified up to isomorphism by the second cohomology group $H^2(G, A)$. Using the explicit description of the cohomology groups $H^n(G, A) = Z^n(G, A)/B^n(G, A)$, see [13] p. 91, an element of $H^2(G, A)$ is represented by a normalised 2-cocycle, i.e. a cocycle $[\,,\,] : G \times G \longrightarrow A$ such that $[g, 1] = [1, g] = 0$. The extension defined by this cocycle is $E = G \times A$ with the following multiplication

$$(g, n) \cdot (h, m) = (gh, [g, h] + n^h + m).$$

For group cohomology exists the Hochschild-Serre-spectral sequence, see [76], p. 195.

Theorem 2.14. *Let F be a group, $N \lhd F$ a normal subgroup and A a F-module. There exists the so called Hochschild-Serre-spectral sequence*

$$E_2^{p,q} = H^p(F/N, H^q(N, A)) \Longrightarrow H^{p+q}(F, A)$$

□

The low degree terms of this spectral sequence give the exact sequence

$$\begin{aligned} 0 \quad \longrightarrow \quad & H^1(F/N, A^N) \longrightarrow H^1(F, A) \longrightarrow H^1(N, A)^{F/N} \\ \longrightarrow \quad & H^2(F/N, A^N) \longrightarrow H^2(F, A), \end{aligned} \tag{9}$$

and the maps in (9) are induced from the inflation and restriction maps

$$\begin{aligned} H^1(F/N, A^N) &\xrightarrow{\text{infl}} H^1(F, A) \\ H^2(F/N, A^N) &\xrightarrow{\text{infl}} H^2(F, A) \end{aligned}$$

$$\begin{aligned} H^1(F, A) &\xrightarrow{\text{restr}} H^1(N, A)^{F/N} \\ H^2(F/N, A^N) &\xrightarrow{\text{restr}} H^2(F, A). \end{aligned}$$

In order to study central extensions of $\mathrm{Sp}(4,\mathbb{Z})$ with coefficients in $\mathbb{Z}$, we will calculate $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$, where $\mathbb{Z}$ is the trivial $\mathrm{Sp}(4,\mathbb{Z})$-module. Let N as in the previous section be the normal subgroup of Br_6 such that

$$\mathrm{Sp}(4,\mathbb{Z}) = \frac{\mathrm{Br}_6}{N}.$$

The associated Hochschild-Serre spectral sequence gives

$$\begin{aligned} 0 \quad &\longrightarrow \quad H^1(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^1(\mathrm{Br}_6,\mathbb{Z}) \xrightarrow{\text{restr}} H^1(N,\mathbb{Z})^{\mathrm{Sp}(4,\mathbb{Z})} \\ &\longrightarrow \quad H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^2(\mathrm{Br}_6,\mathbb{Z}). \end{aligned} \tag{10}$$

As $\mathbb{Z}$ is the trivial $\mathrm{Sp}(4,\mathbb{Z})$-modul, $H^1(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z}) = \mathrm{Hom}(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$ and $H^1(\mathrm{Br}_6,\mathbb{Z}) = \mathrm{Hom}(\mathrm{Br}_6,\mathbb{Z})$. Furthermore a simple calculation shows that $H^1(N,\mathbb{Z})^{\mathrm{Sp}(4,\mathbb{Z})} = \mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}$, where Br_6 acts by

$$(\varphi \cdot g)(x) = \varphi(g^{-1}xg),$$

on $\varphi \in \mathrm{Hom}(N,\mathbb{Z})$.

Proposition 2.15. *Let $\mathbb{Z}$ be the trivial $\mathrm{Sp}(4,\mathbb{Z})$-modul. Then*

$$H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z}) \simeq \frac{\mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}}{\mathrm{Hom}(Br_6,\mathbb{Z})} \tag{11}$$

Proof: In order to prove Proposition 2.15, we need the following lemma.

Lemma 2.16. *The abelianisation $\mathrm{Sp}(4,\mathbb{Z})_{ab}$ of $\mathrm{Sp}(4,\mathbb{Z})$ is $\mathbb{Z}_2$.*

Proof: It is clear from the braid relations that the abelianisation of Br_6 is $\mathbb{Z}$. Now the fact that κ and γ have degree 10 and 12 respectively, whereas π has degree zero, implies $\mathrm{Sp}(4,\mathbb{Z})_{ab} \simeq \mathbb{Z}_2$.

□

Thus $H_1(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z}) = \mathrm{Sp}(4,\mathbb{Z})_{ab} = \mathbb{Z}_2$. Therefore it follows from the universal coefficient theorem, that $H^1(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$ is trivial. For $n < 12$ Arnol'd shows in [2], that

$$\begin{aligned} H^1(\mathrm{Br}_n,\mathbb{Z}) &= \mathbb{Z} \\ H^2(\mathrm{Br}_n,\mathbb{Z}) &= 0. \end{aligned}$$

Now the Hochschild-Serre spectral sequence (10) implies the proposition.

□

Remark 2.17. Recall the degree homomorphism $\deg : \mathrm{Br}_6 \longrightarrow \mathbb{Z}$ from Remark 2.7. Note that $\mathrm{Hom}(\mathrm{Br}_6,\mathbb{Z}) = \mathbb{Z}$ is generated by the degree homomorphism and that the inclusion $\mathrm{Hom}(\mathrm{Br}_6,\mathbb{Z}) \subset \mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}$ is given by restriction.

We realize the extension defined by a Br_6-invariant homomorphism $\varphi : N \longrightarrow \mathbb{Z}$ explicitly as follows. Denote the inclusion

$$\varphi : N \hookrightarrow \mathrm{Br}_6 \times \mathbb{Z};\ g \mapsto (g, -\varphi(g)).$$

by the same letter φ. Then

Proposition 2.18. *Let φ be an element of $\mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}$. Then the group*

$$G_\varphi := \frac{\mathrm{Br}_6 \times \mathbb{Z}}{\varphi(N)}.$$

fits into an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow G_\varphi \longrightarrow \mathrm{Sp}(4,\mathbb{Z}) \longrightarrow 1$$

and this extension is isomorphic to the one that corresponds to φ under the isomorphism (11).

Proof: We first describe the isomorphism (11). Let φ denote an element of $\mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}$. Choose a representative $t_g \in \mathrm{Br}_6$ for each $g \in \mathrm{Sp}(4,\mathbb{Z})$ such that $t_e = e$ and define $r(g,h) \in \mathrm{Br}_6$ by $t_g \cdot t_h = t_{gh}\, r(g,h)$. Then $\overline{\varphi}(g,h) := \varphi(r(g,h))$ defines a normalised 2-cocycle and thus an extension class. $\overline{\varphi}$ is a coboundary iff φ is induced by an element of $\mathrm{Hom}(\mathrm{Br}_6,\mathbb{Z})$. This is the isomorphism (11).
Consider now the group

$$G_\varphi := \frac{\mathrm{Br}_6 \times \mathbb{Z}}{\varphi(N)}$$

Let $(x, n), (y, m)$ be elements of G_φ and denote the elements in $\mathrm{Sp}(4, \mathbb{Z})$ corresponding to x and y by g and h respectively. Then there are $k, l \in N$ such that $x = t_g k, y = t_h l \in \mathrm{Br}_6$. Multiplication in G_φ gives

$$\begin{aligned}(x, n) \cdot (y, m) &= (xy, n + m) \\ &= (t_{gh} r(g, h), n + m + \varphi(kl)) \\ &= (t_{gh}, \overline{\varphi}(g, h) + n + m + \varphi(kl))\end{aligned}$$

Thus G_φ is isomorphic to the extension defined by the cocycle $\overline{\varphi}$.

□

Remark 2.19. Conversely given an extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow E \longrightarrow \mathrm{Sp}(4, \mathbb{Z}) \longrightarrow 1$$

that represents a cohomology class $c \in H^2(\mathrm{Sp}(4, \mathbb{Z}), \mathbb{Z})$, one gets the corresponding element of $\mathrm{Hom}(N, \mathbb{Z})^{\mathrm{Br}_6}$ in the following way. Because of $H^2(\mathrm{Br}_6, \mathbb{Z}) = 0$, the pullback of this extension under $q : \mathrm{Br}_6 \longrightarrow \mathrm{Sp}(4, \mathbb{Z})$ must be trivial, i.e.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathrm{Br}_6 \times \mathbb{Z} \arrow[r] \arrow[d, "q'"] & \mathrm{Br}_6 \arrow[r] \arrow[d, "q"] & 1 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & E \arrow[r] & \mathrm{Sp}(4, \mathbb{Z}) \arrow[r] & 1 \end{tikzcd}$$

A choice of a splitting $\sigma = \mathrm{id} \times \alpha : \mathrm{Br}_6 \longrightarrow \mathrm{Br}_6 \times \mathbb{Z}$ determines a homomorphism $\tilde{q} : \mathrm{Br}_6 \longrightarrow E$ defined by $\tilde{q}(b) = q'(\sigma(b))$. Its restriction to N yields a Br_6-invariant homomorphism $\varphi := \tilde{q}_{|N} : N \longrightarrow \mathbb{Z}$, that is well defined up to an element of $\mathrm{Hom}(\mathrm{Br}_6, \mathbb{Z})$. Note that the choice $\alpha \equiv 0$ gives $\ker q' = \{(b, n) | b \in N,\ n = -\varphi(b)\}$. Thus this gives the correct element of $\mathrm{Hom}(N, \mathbb{Z})^{\mathrm{Br}_6} / \mathrm{Hom}(\mathrm{Br}_6, \mathbb{Z})$.

Remark 2.20. $\ker(G_\varphi \longrightarrow \mathrm{Sp}(4, \mathbb{Z})) = \frac{N \times \mathbb{Z}}{\varphi(N)}$ and the isomorphism $\frac{N \times \mathbb{Z}}{\varphi(N)} \simeq \mathbb{Z}$ is induced by $(r, n) \mapsto \varphi(r) + n$. Note that in case $\varphi : N \longrightarrow \mathbb{Z}$ is surjective, $\ker \varphi \lhd \mathrm{Br}_6$ and

$$G_\varphi := \frac{\mathrm{Br}_6}{\ker \varphi}.$$

In this case the isomorphism $\frac{N}{\ker \varphi} \simeq \mathbb{Z}$ is induced by φ.

2.2.2. *The standard generators of* $\mathrm{Sp}(4,\mathbb{Z})$. We interpret $(\mathbb{Z}^4,\omega)$ as $H_1(\Sigma,\mathbb{Z})$ of a surface Σ of genus two with the intersection pairing. The standard symplectic basis of $H_1(\Sigma,\mathbb{Z})$ is $(\alpha_1,\beta_1,\alpha_2,\beta_2)$, where these classes correspond to the curves S_2, S_1, S_4, S_5 in Figure 4 in that order. According to Theorem 2.9 the standard generators of $\mathrm{Sp}(4,\mathbb{Z})$ are then

$$\begin{aligned} a_1 &= t_{\beta_1} \\ a_2 &= t_{\alpha_1} \\ a_3 &= t_{\beta_1+\beta_2} \\ a_4 &= t_{\alpha_2} \\ a_5 &= t_{\beta_2}. \end{aligned}$$

In the following sections we will discuss two natural central extensions of $\mathrm{Sp}(4,\mathbb{Z})$.

2.3. **The universal cover of the symplectic group** $\mathrm{Sp}(4,\mathbb{Z})$**.** Consider the standard symplectic lattice $(\mathbb{Z}^4,\omega)$ and the corresponding symplectic vector space $(\mathbb{R}^4,\omega)$. The maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ is isomorphic to $U(2)$ and the inclusion $\mathrm{U}(2)\subset \mathrm{Sp}(4,\mathbb{R})$ is a homotopy equivalence. Furthermore the complex determinant $\det_{\mathbb{C}}:\mathrm{U}(2)\longrightarrow S^1$ induces an isomorphism $\pi_1(\mathrm{U}(2))\simeq\mathbb{Z}$. Thus the universal cover $\widetilde{\mathrm{Sp}}(4,\mathbb{R})$ of the real symplectic group is naturally a central extension

$$0\longrightarrow\mathbb{Z}\longrightarrow\widetilde{\mathrm{Sp}}(4,\mathbb{R})\longrightarrow\mathrm{Sp}(4,\mathbb{R})\longrightarrow 1. \tag{12}$$

The pullback of this extension under the inclusion $\mathrm{Sp}(4,\mathbb{Z})\hookrightarrow\mathrm{Sp}(4,\mathbb{R})$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \widetilde{\mathrm{Sp}}(4,\mathbb{Z}) \arrow[r] \arrow[d, hook] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[r] \arrow[d, hook] & 1 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \widetilde{\mathrm{Sp}}(4,\mathbb{R}) \arrow[r] & \mathrm{Sp}(4,\mathbb{R}) \arrow[r] & 1 \end{tikzcd} \tag{13}$$

yields a central extension of the group $\mathrm{Sp}(4,\mathbb{Z})$. The group $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ is called the universal cover of $\mathrm{Sp}(4,\mathbb{Z})$. Universal covers of symplectic groups are also discussed in [9]. The group $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ is canonically isomorphic to the relative homotopy group

$$\pi_1(\mathrm{Sp}(4,\mathbb{R}),\mathrm{Sp}(4,\mathbb{Z}),id).$$

As a set the latter consists in homotopy classes of paths in $\mathrm{Sp}(4,\mathbb{R})$ that join id to an element of $\mathrm{Sp}(4,\mathbb{Z})$. This set carries a group structure in the following way.

Let $[\gamma]$ and $[\gamma']$ be two homotopy classes with representatives

$$\begin{array}{rcl} \gamma, \gamma' : [0,1] & \longrightarrow & \mathrm{Sp}(4,\mathbb{R}) \\ s & \mapsto & \gamma(s), \gamma'(s) \end{array}$$

then $[\gamma'] \cdot [\gamma]$ is the class represented by

$$\gamma' \cdot \gamma(s) := \left\{ \begin{array}{c} \gamma(2s) \qquad ; s \in [0, \frac{1}{2}] \\ \gamma'(2s-1) \circ \gamma(1) \, ; s \in [\frac{1}{2}, 1] \end{array} \right\}.$$

The neutral element is the class of the constant path and the inverse of an element $[\gamma]$ is represented by

$$\gamma^{-1}(s) := \gamma(1-s) \circ \gamma(1)^{-1}.$$

The fundamental group $\pi_1(\mathrm{Sp}(4,\mathbb{R}))$ is naturally a subgroup of $\pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id)$.

Lemma 2.21. *The group $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ is canonically isomorphic to $\pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id)$. Under this isomorphism the extension (13) coincides with the natural exact sequence*

$$0 \longrightarrow \pi_1(\mathrm{Sp}(4,\mathbb{R})) \longrightarrow \pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id) \longrightarrow \mathrm{Sp}(4,\mathbb{Z}) \longrightarrow 1$$

where the third arrow is evaluation map $[\gamma] \mapsto \gamma(1)$.

Proof: An element $[\gamma]$ of $\pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id)$ is represented by a path γ in $\mathrm{Sp}(4,\mathbb{R})$. γ has a unique lift $\widetilde{\gamma}$ in $\widetilde{\mathrm{Sp}}(4,\mathbb{R}) \longrightarrow \mathrm{Sp}(4,\mathbb{R})$ with starting point the neutral element $1 \in \widetilde{\mathrm{Sp}}(4,\mathbb{R})$. Associating to $[\gamma]$ the endpoint of $\widetilde{\gamma}$ gives a well defined group homomorphism $\Psi : \pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id) \longrightarrow \widetilde{\mathrm{Sp}}(4,\mathbb{Z})$. Conversely let x be an element of $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$. Then chose a path in $\widetilde{\mathrm{Sp}}(4,\mathbb{R})$ that joins 1 with x. Projecting to $\mathrm{Sp}(4,\mathbb{R})$ yields a well defined homotopy class in $\pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id)$. The two constructions are obviously inverse to each other, so Ψ is indeed an isomorphism. The second statement of the lemma follows from the description of Ψ.

$\square$

From now on we identify $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ with $\pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id)$.

To a transvection t_v in $\mathrm{Sp}(4,\mathbb{Z})$ is naturally associated the one parameter subgroup

$$s \mapsto [x \mapsto x + s \cdot \omega(v,x)v]$$

of $\mathrm{Sp}(4,\mathbb{R})$. We call the restriction $[0,1] \ni s \mapsto t_{\sqrt{s}v}$ the natural path the joins id to the transvection t_v. This path defines an element of $\pi_1(\mathrm{Sp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{Z}), id)$ that is a lift of t_v.

Definition 2.22. *By a lift of $T \in \mathrm{Sp}(4,\mathbb{Z})$ in a central extension $\mathbb{Z} \longrightarrow E \longrightarrow \mathrm{Sp}(4,\mathbb{Z})$ we mean inverse image of T under $E \longrightarrow \mathrm{Sp}(4,\mathbb{Z})$.*

Definition 2.23. *Let $t \in \mathrm{Sp}(4,\mathbb{Z})$ be a transvection. The element $\widetilde{t}$ of $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ that corresponds to the natural path from id to t will be called the* distinguished lift of t. By the distinguished lift of a product $\prod_i t_i$ of transvections is meant the product $\prod_i \widetilde{t}_i$ of the distinguished lifts.

Now that we have described the universal cover of $\mathrm{Sp}(4,\mathbb{Z})$, we ask, which element of

$$\frac{\mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}}{\mathrm{Hom}(\mathrm{Br}_6,\mathbb{Z})}$$

corresponds to the extension (13)? Consider the pullback of this extension under the homomorphism $q : \mathrm{Br}_6 \longrightarrow \mathrm{Sp}(4,\mathbb{Z})$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathrm{Br}_6 \times_{\mathrm{Sp}(4,\mathbb{Z})} \widetilde{\mathrm{Sp}}(4,\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{pr}_2"] & \mathrm{Br}_6 \arrow[r] \arrow[d, "q"] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \widetilde{\mathrm{Sp}}(4,\mathbb{Z}) \arrow[r, "\pi"] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[r] & 1
\end{tikzcd} \tag{14}$$

The sequence (14) splits as $H^2(\mathrm{Br}_6,\mathbb{Z})$ is trivial. For the rest of the section we denote the standard generators of Br_6 by $a_i^{\mathrm{Br}_6}$, $i = 1, ..., 5$ and by a_i the corresponding elements of $\mathrm{Sp}(4,\mathbb{Z})$.

Lemma 2.24. *There exists a splitting*

$$\sigma : \mathrm{Br}_6 \longrightarrow \mathrm{Br}_6 \times_{\mathrm{Sp}(4,\mathbb{Z})} \widetilde{\mathrm{Sp}}(4,\mathbb{Z})$$

of the sequence (14), such that $a_i^{\mathrm{Br}_6} \overset{\sigma}{\mapsto} (a_i^{\mathrm{Br}_6}, \widetilde{a}_i)$.

Proof: It suffices to show that the distinguished lifts $\widetilde{a}_i$ satisfy the braid relations. In this proof we use the following notation. For any path $\gamma : [0,1] \longrightarrow \mathrm{Sp}(4,\mathbb{R})$ with $\gamma(0) = id$ we denote by $\widetilde{\gamma}$ the unique lift with $\widetilde{\gamma}(0) = 1 \in \mathrm{Sp}(4,\mathbb{R})$.

Consider first that case $i, j \in \{1, ..., 5\}$ such that $|i - j| \geq 2$. Denote by γ_k the natural path in $\mathrm{Sp}(4,\mathbb{R})$ that joins id to a_k and by $\widetilde{\gamma}_k$ the unique lift of γ_k with starting point $1 \in \widetilde{\mathrm{Sp}}(4,\mathbb{R})$. As $a_i a_j = a_j a_i$, the two paths $\gamma_i \cdot \gamma_j$ and $\gamma_j \cdot \gamma_i$ have the same endpoint. A homotopy relative to $\{id, a_i a_j\} \subset \mathrm{Sp}(4,\mathbb{R})$ between them is (up to reparametrisation) given by

$$h_t(s) := \left\{ \begin{array}{cl} \gamma_j(3s(1-t)) & ; s \in [0, \frac{1}{3}] \\ \gamma_i(3s-1) \circ \gamma_j(1-t) & ; s \in [\frac{1}{3}, \frac{2}{3}] \\ \gamma_i(1) \circ \gamma_j(3t(s-1)+1) & ; s \in [\frac{2}{3}, 1] \end{array} \right\}$$

Thus it follows that the paths $\widetilde{\gamma_i \cdot \gamma_j}$ and $\widetilde{\gamma_j \cdot \gamma_i}$ have the same endpoint. This in turn implies

$$\begin{aligned} \widetilde{a}_i \widetilde{a}_j &= \widetilde{\gamma}_i(1)\widetilde{\gamma}_j(1) \\ &= \widetilde{\gamma_i \cdot \gamma_j}(1) \\ &= \widetilde{\gamma_j \cdot \gamma_i}(1) \\ &= \widetilde{\gamma}_j(1)\widetilde{\gamma}_i(1) \\ &= \widetilde{a}_j \widetilde{a}_i. \end{aligned}$$

Consider now the case $i \in \{1, ..., 4\}$ and $j = i+1$. As $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ the two paths $\gamma_i \cdot \gamma_{i+1} \cdot \gamma_i$ and $\gamma_{i+1} \cdot \gamma_i \cdot \gamma_{i+1}$ have the same endpoint. We claim that a homotopy relative to $\{id, a_i a_{i+1} a_i\} \subset \mathrm{Sp}(4,\mathbb{R})$ between the two paths is (up to reparametrisation) given by

$$h_t(s) := \left\{ \begin{array}{cl} \gamma_i(4s(1-t)) & ; s \in [0, \frac{1}{4}] \\ \gamma_{i+1}(4s-1)\gamma_i(1-t) & ; s \in [\frac{1}{4}, \frac{1}{2}] \\ \gamma_i(4s-2)\gamma_{i+1}(1)\gamma_i(1-t) & ; s \in [\frac{1}{2}, \frac{3}{4}] \\ \gamma_{i+1}((4s-3)t)\gamma_i(1)\gamma_{i+1}(1)\gamma_i(1-t) & ; s \in [\frac{3}{4}, 1] \end{array} \right\}.$$

It suffices to check that $h_t(1) = a_{i+1} a_i a_{i+1}$ for all $t \in [0,1]$. For $i = 1, ..., 4$ there are $u, v \in \mathbb{Z}^4$ with $\omega(u,v) = 1$ such that $a_i = t_u$ and $a_{i+1} = t_v$. Let $t \in [0,1]$ and $x \in \mathbb{Z}^4$, then

$$\begin{aligned} h_t(1)x &= \gamma_{i+1}(t)\gamma_i(1)\gamma_{i+1}(1)\gamma_i(1-t)x \\ &= t_{\sqrt{t}v} t_u t_v t_{\sqrt{1-t}u} x \\ &= x + (1-t)\omega(v,x)v + \omega(u,x)u + (1-t)\omega(v,x)u + \omega(v,x)v - \omega(u,x)v \\ &\quad -(1-t)\omega(v,x)v + t\omega(u,x)u + t\omega(u,v)\omega(v,x)u - t\omega(u,x)u \\ &= x + \omega(u,x)u + \omega(v,x)u + \omega(v,x)v - \omega(u,x)v. \end{aligned}$$

The last expression is independent of t and therefore $h_t(1) = a_{i+1} a_i a_{i+1}$ for all $t \in [0,1]$. Thus it follows that the paths $\widetilde{(\gamma_i \cdot \gamma_{i+1} \cdot \gamma_i)}$ and $\widetilde{(\gamma_{i+1} \cdot \gamma_i \cdot \gamma_{i+1})}$ have the same endpoint, which in turn implies

$$\begin{aligned}
\widetilde{a}_i\widetilde{a}_{i+1}\widetilde{a}_i &= \widetilde{\gamma}_i(1)\widetilde{\gamma}_{i+1}(1)\widetilde{\gamma}_i(1)\\
&= (\widetilde{\gamma_i\cdot\gamma_{i+1}\cdot\gamma_i})(1)\\
&= (\widetilde{\gamma_{i+1}\cdot\gamma_i\cdot\gamma_{i+1}})(1)\\
&= \widetilde{\gamma}_{i+1}(1)\widetilde{\gamma}_i(1)\widetilde{\gamma}_{i+1}(1)\\
&= \widetilde{a}_{i+1}\widetilde{a}_i\widetilde{a}_{i+1}.
\end{aligned}$$

□

Proposition 2.25. *There is a unique homomorphism $\widetilde{q}:\mathrm{Br}_6\longrightarrow\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ such that $\pi\circ\widetilde{q}=q$ and such that $\widetilde{q}(a_i^{\mathrm{Br}_6})=\widetilde{a}_i$, i.e. $\widetilde{q}$ maps $a_i^{\mathrm{Br}_6}$ to the canonical lift of a_i. In particular $\widetilde{q}$ restricts to an invariant homomorphism $\widetilde{q}_{|N}:N\longrightarrow\mathbb{Z}$ and*

$$\widetilde{\mathrm{Sp}}(4,\mathbb{Z})=\frac{\mathrm{Br}_6\times_{\mathrm{Sp}(4,\mathbb{Z})}\widetilde{\mathrm{Sp}}(4,\mathbb{Z})}{\widetilde{q}(N)}.$$

Proof: According to Lemma 2.24 $\widetilde{q}:=\mathrm{pr}_2\circ\sigma$ is a homomorphism that satisfies the requirements $\pi\circ\widetilde{q}=q$ and $\widetilde{q}(a_i^{\mathrm{Br}_6})=\widetilde{a}_i$. As $N=\ker q$ the restriction of $\widetilde{q}$ to N maps into $\mathbb{Z}\subset\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$.

□

We will use this to arrive at an explicit description of $\widetilde{q}_{|N}:N\longrightarrow\mathbb{Z}$.

Definition 2.26. *Let $(H_{\mathbb{R}},\omega)$ be a symplectic vector space. By the Lagrangian Grassmannian Λ_ω, we mean the Grassmannian of oriented Lagrangian subspaces of $H_{\mathbb{R}}$.*

The Lagrangian Grassmannian can be identified with the homogeneous space $\mathrm{U}(g)/\mathrm{SO}(g)$ as follows, see [9]. Choose a compatible complex structure I on $H_{\mathbb{R}}$, i.e. $g(x,y):=\omega(I(x),y)$ is a positive definite, symmetric form. $h:=g+i\omega$ is then a positive definite, hermitian form on $(H_{\mathbb{R}},I)$. Now $\mathrm{U}(g)\simeq\mathrm{U}(H_{\mathbb{R}},I,h)$ is naturally a subgroup of $\mathrm{Sp}(H_{\mathbb{R}},\omega)$. It acts transitively on Λ_ω with isotropy group $\mathrm{SO}(H_{\mathbb{R}},g)$. Thus $\Lambda_\omega\simeq\mathrm{U}(g)/\mathrm{SO}(g)$. The complex determinant induces a map $\det_{\mathbb{C}}:\Lambda_\omega\longrightarrow S^1$, which in turn induces an isomorphism $\pi_1(\Lambda_\omega)=\mathbb{Z}$.

Consider $(\mathbb{R}^4,\omega)$. The symplectic group $\mathrm{Sp}(4,\mathbb{R})$ acts transitively on Λ_ω. And this action lifts to a transitive action of $\widetilde{\mathrm{Sp}}(4,\mathbb{R})$ on the universal cover $\widetilde{\Lambda_\omega}$ of Λ_ω. Fix an element Λ_0 of Λ_ω and let t_v be a transvection in $\mathrm{Sp}(4,\mathbb{Z})$. Then the natural path $s\mapsto t_{\sqrt{s}v}$ leads to a path $\Lambda_s=t_{\sqrt{s}v}\cdot\Lambda_0$ in Λ_ω.

Proposition 2.27. *Let $\prod_i t_{v_i} = 1$, for $v_i \in \mathbb{Z}^4$ be a relation in* $\mathrm{Sp}(4,\mathbb{Z})$*. Then the distinguished lift $\prod_i \widetilde{t}_{v_i}$ lies in $\pi_1(\mathrm{Sp}(4,\mathbb{R})) = \mathbb{Z}$ and the integer*

$$\prod_i \widetilde{t}_{v_i} \in \mathbb{Z}$$

corresponds to the class of $\prod_i \widetilde{t}_{v_i} \cdot \Lambda_0$ in $\pi_1(\Lambda_\omega) = \mathbb{Z}$.

Proof: The choice of an element $\Lambda_0 \in \Lambda_\omega$ defines maps $\mathrm{Sp}(4,\mathbb{R}) \longrightarrow \Lambda_\omega$ and $\mathrm{U}(2) \longrightarrow \Lambda_\omega$. Together with the inclusion $\mathrm{U}(2) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$ these maps induce a commutative diagram

$$\begin{tikzcd}
\pi_1(\mathrm{Sp}(4,\mathbb{R})) \arrow[rr] & & \pi_1(\Lambda_\omega) \\
& \pi_1(\mathrm{U}(2)) \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

This proves the claim.

$\square$

Recall that

$$N = \langle \kappa, \pi, \gamma \rangle.$$

Lemma 2.28. *The values of $\widetilde{q}_{|N} : N \longrightarrow \mathbb{Z}$ on the (normal) generators of N are*

$$\widetilde{q}(\kappa) = \widetilde{q}(\gamma) = -1,\ \widetilde{q}(\pi) = 0$$

Proof: Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be the standard symplectic basis in $\mathbb{Z}^4$, identify $\mathbb{R}^4$ with $\mathbb{C}^2$ via $\alpha_i = e_i$, $\beta_i = ie_i$ and fix the Lagrangian subspace $\Lambda_0 = \mathrm{span}_{\mathbb{R}}(\alpha_1, \alpha_2)$. As in Section 2.2.2 the generators $a_1, ..., a_5$ of $\mathrm{Sp}(4,\mathbb{Z})$ are $a_1 = t_{\beta_1}$, $a_2 = t_{\alpha_1}$, $a_3 = t_{\beta_1+\beta_2}$, $a_4 = t_{\alpha_2}$, $a_5 = t_{\beta_2}$.

Consider a path $\varphi : [0,1] \longrightarrow \mathrm{Sp}(4,\mathbb{R})$ and denote by $det\varphi$ the path $[0,1] \ni s \mapsto \det_{\mathbb{C}}(\varphi_s \cdot \Lambda_0) \in S^1$. We want to calculate homotopy classes of such paths. For $A \in \mathrm{Sp}(4,\mathbb{R})$ the two vectors $A\alpha_1, A\alpha_2 \in \mathbb{C}^2$ determine the element $L := A \cdot \Lambda_0$ in Λ_ω. If $v_1, v_2 \in \mathbb{C}^2$ is another frame for L, then $\det_{\mathbb{C}}(v_1, v_2) = \det_{\mathbb{C}}(A\alpha_1, A\alpha_2) \det_{\mathbb{C}}(S)$, where $S \in \mathrm{GL}^+(2,\mathbb{R})$ is the base change. As $\det_{\mathbb{C}}(L)$ has been defined above as the determinant of a unitary Lagrangian frame (v_1, v_2), $\det_{\mathbb{C}}(L) = \det_{\mathbb{C}}(A\alpha_1, A\alpha_2)/|\det_{\mathbb{C}}(A\alpha_1, A\alpha_2)|$. Thus

$$det\varphi_s = \frac{\det_{\mathbb{C}}(\varphi_s\alpha_1, \varphi_s\alpha_2)}{|\det_{\mathbb{C}}(\varphi_s\alpha_1, \varphi_s\alpha_2)|}$$

and the path $det\varphi : [0,1] \longrightarrow S^1$ is homotopic to the path $\det_{\mathbb{C}}(\varphi_s\alpha_1, \varphi_s\alpha_2)$ in $\mathbb{C}^*$.

Consider first $\gamma \in N$. In $\mathrm{Sp}(4,\mathbb{Z})$ this is the relation $(a_1a_2)^6 = 1$. The distinguished lift in $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ is the loop $(\widetilde{a}_1\widetilde{a}_2)^6$. Its action is

	(w_1, w_2)		$\det_{\mathbb{C}}(w_1, w_2)$
	(α_1, α_2)	(e_1, e_2)	1
$\xrightarrow{a_2}$	(α_1, α_2)	(e_1, e_2)	1
$\xrightarrow{a_1}$	$(\alpha_1 - \beta_1, \alpha_2)$	$((1-i)e_1, e_2)$	$1-i$
$\xrightarrow{a_2}$	$(-\beta_1, \alpha_2)$	$(-ie_1, e_2)$	$-i$
$\xrightarrow{a_1}$	$(-\beta_1, \alpha_2)$	$(-ie_1, e_2)$	$-i$
$\xrightarrow{a_2}$	$(-\alpha_1 - \beta_1, \alpha_2)$	$((-1-i)e_1, e_2)$	$-1-i$
$\xrightarrow{a_1}$	$(-\alpha_1, \alpha_2)$	$(-e_1, e_2)$	-1
$\xrightarrow{a_2}$	$(-\alpha_1, \alpha_2)$	$(-e_1, e_2)$	-1
$\xrightarrow{a_1}$	$(-\alpha_1 + \beta_1, \alpha_2)$	$((-1+i)e_1, e_2)$	$-1+i$
$\xrightarrow{a_2}$	(β_1, α_2)	(ie_1, e_2)	i
$\xrightarrow{a_1}$	(β_1, α_2)	(ie_1, e_2)	i
$\xrightarrow{a_2}$	$(\alpha_1 + \beta_1, \alpha_2)$	$((1+i)e_1e_2)$	$1+i$
$\xrightarrow{a_1}$	(α_1, α_2)	(e_1, e_2)	1

It follows that the path $\det_{\mathbb{C}}((\widetilde{a}_1\widetilde{a}_2)^6 \cdot \Lambda_0)$ is homotopic to the path $[0,1] \ni s \mapsto e^{-2\pi i s}$ and therefore that $\widetilde{q}(\gamma) = -1$.

As π is a commutator, the corresponding path in Λ_ω is homotopic to the constant path. Thus $\widetilde{q}(\pi) = 0$.

Consider now $\kappa \in N$. In $\mathrm{Sp}(4,\mathbb{Z})$ this is the relation $(a_1a_2a_3a_4)^{10}(a_5a_1a_2a_3a_4)^{-6}$. We treat $(a_5a_1a_2a_3a_4)^6$ first.

	(w_1, w_2)		$\det_{\mathbb{C}}$
	(α_1, α_2)	(e_1, e_2)	1
$\xrightarrow{a_4}$	(α_1, α_2)	(e_1, e_2)	1
$\xrightarrow{a_3}$	$(\alpha_1 - \beta_1 - \beta_2, -\beta_1 + \alpha_2 - \beta_2)$	$((1-i)e_1 - ie_2, -ie_1 + (1-i)e_2)$	$1-2i$
$\xrightarrow{a_2}$	$(-\beta_1 - \beta_2, -\alpha_1 - \beta_1 + \alpha_2 - \beta_2)$	$(-ie_1 - ie_2, (-1-i)e_1 + (1-i)e_2)$	$-2i$
$\xrightarrow{a_1}$	$(-\beta_1 - \beta_2, -\alpha_1 + \alpha_2 - \beta_2)$	$(-ie_1 - ie_2, -e_1 + (1-i)e_2)$	$-1-2i$
$\xrightarrow{a_5}$	$(-\beta_1 - \beta_2, -\alpha_1 + \alpha_2 - 2\beta_2)$	$(-ie_1 - ie_2, -e_1 + (1-2i)e_2)$	$-2-2i$
$\xrightarrow{a_4}$	$(-\beta_1 - \alpha_2 - \beta_2, -\alpha_1 - \alpha_2 - 2\beta_2)$	$(-ie_1 + (-1-i)e_2, -e_1 + (-1-2i)e_2)$	-3
$\xrightarrow{a_3}$	$(-\alpha_2, -\alpha_1 + 2\beta_1 - \alpha_2)$	$(-e_2, (-1+2i)e_1 - e_2)$	$-1+2i$
$\xrightarrow{a_2}$	$(-\alpha_2, \alpha_1 + 2\beta_1 - \alpha_2)$	$(-e_2, (1+2i)e_1 - e_2)$	$1+2i$
$\xrightarrow{a_1}$	$(-\alpha_2, \alpha_1 + \beta_1 - \alpha_2)$	$(-e_2, (1+i)e_1 - e_2)$	$1+i$
$\xrightarrow{a_5}$	$(-\alpha_2 + \beta_2, \alpha_1 + \beta_1 - \alpha_2 + \beta_2)$	$((-1+i)e_2, (1+i)e_1 + (-1+i)e_2)$	2
$\xrightarrow{a_4}$	$(\beta_2, \alpha_1 + \beta_1 + \beta_2)$	$(ie_1, (1+i)e_1 + ie_2)$	$1-i$
$\xrightarrow{a_3}$	(β_2, α_1)	(ie_2, e_1)	$-i$
$\xrightarrow{a_2}$	(β_2, α_1)	(ie_2, e_1)	$-i$
$\xrightarrow{a_1}$	$(\beta_2, \alpha_1 - \beta_1)$	$(ie_2, (1-i)e_1)$	$-1-i$
$\xrightarrow{a_5}$	$(\beta_2, \alpha_1 - \beta_1)$	$(ie_2, (1-i)e_1)$	$-1-i$
$\xrightarrow{a_4}$	$(-\alpha_2 + \beta_2, \alpha_1 - \beta_1)$	$((1+i)e_2, (1-i)e_1)$	$-2i$
$\xrightarrow{a_3}$	$(\alpha_2 - \beta_1, \alpha_1 - 2\beta_1 - \beta_2)$	$(-ie_1 + e_2, (1-2i)e_1 - ie_2)$	$-2+2i$
$\xrightarrow{a_2}$	$(-\alpha_1 - \beta_1 + \alpha_2, -\alpha_1 - 2\beta_1 - \beta_2)$	$((-1-i)e_1 + e_2, (-1-2i)e_1 - ie_2)$	$3i$
$\xrightarrow{a_1}$	$(-\alpha_1 + \alpha_2, -\alpha_1 - \beta_1 - \beta_2)$	$(-e_1 + e_2, (-1-i)e_1 - ie_2)$	$1+2i$
$\xrightarrow{a_5}$	$(-\alpha_1 + \alpha_2 - \beta_2, -\alpha_1 - \beta_1 - \beta_2)$	$(-e_1 + (1-i)e_2, (-1-i)e_1 - ie_2)$	$2+i$
$\xrightarrow{a_4}$	$(-\alpha_1 - \beta_2, -\alpha_1 - \beta_1 - \alpha_2 - \beta_2)$	$(-e_1 - ie_2, (-1-i)e_1 + (-1-i)e_2)$	2
$\xrightarrow{a_3}$	$(-\alpha_1 + \beta_1, -\alpha_1 + \beta_1 - \alpha_2 + \beta_2)$	$((-1+i)e_1, (-1+i)e_1 + (-1+i)e_2)$	$-2i$
$\xrightarrow{a_2}$	$(\beta_1, \beta_1 - \alpha_2 + \beta_2)$	$(ie_1, ie_1 + (-1+i)e_2)$	$-1-i$
$\xrightarrow{a_1}$	$(\beta_1, \beta_1 - \alpha_2 + \beta_2)$	$(ie_1, ie_1 + (-1+i)e_2)$	$-1-i$
$\xrightarrow{a_5}$	$(\beta_1, \beta_1 - \alpha_2 + 2\beta_2)$	$(ie_1, ie_1 + (-1+2i)e_2)$	$-2-i$
$\xrightarrow{a_4}$	$(\beta_1, \beta_1 + \alpha_2 + 2\beta_2)$	$(ie_1, ie_1 + (1+2i)e_2)$	$-2+i$
$\xrightarrow{a_3}$	$(\beta_1, \alpha_2 + \beta_2)$	$(ie_1, (1+i)e_2)$	$-1+i$
$\xrightarrow{a_2}$	$(\alpha_1 + \beta_1, \alpha_2 + \beta_2)$	$((1+i)e_1, (1+i)e_2)$	$2i$
$\xrightarrow{a_1}$	$(\alpha_1, \alpha_2 + \beta_2)$	$(e_1, (1+i)e_2)$	$1+i$
$\xrightarrow{a_5}$	(α_1, α_2)	(e_1, e_2)	1

It follows that $\det_{\mathbb{C}}((\widetilde{a}_5\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4)^{-6} \cdot \Lambda_0)$ is homotopic to the path $[0,1] \ni s \mapsto e^{6\pi i s}$.

For $(a_1a_2a_3a_4)^5$ on the other hand we get.

	(w_1,w_2)		$\det_{\mathbb{C}}$
	(α_1,α_2)	(e_1,e_2)	1
$\overset{a_4}{\to}$	(α_1,α_2)	(e_1,e_2)	1
$\overset{a_3}{\to}$	$(\alpha_1-\beta_1-\beta_2,-\beta_1+\alpha_2-\beta_2)$	$((1-i)e_1-ie_2,-ie_1+(1-i)e_2)$	$1-2i$
$\overset{a_2}{\to}$	$(-\beta_1-\beta_2,-\alpha_1-\beta_1+\alpha_2-\beta_2)$	$(-ie_1-ie_2,(-1-i)e_1+(1-i)e_2)$	$-2i$
$\overset{a_1}{\to}$	$(-\beta_1-\beta_2,-\alpha_1+\alpha_2-\beta_2)$	$(-ie_1-ie_2,-e_1+(1-i)e_2)$	$-1-2i$
$\overset{a_4}{\to}$	$(-\beta_1-\alpha_2-\beta_2,-\alpha_1-\beta_2)$	$(-ie_1+(-1-i)e_2,-e_1-ie_2)$	$-2-i$
$\overset{a_3}{\to}$	$(-\alpha_2,-\alpha_1+\beta_1)$	$(-e_2,(-1+i)e_1)$	$-1+i$
$\overset{a_2}{\to}$	$(-\alpha_2,\beta_1)$	$(-e_2,ie_1)$	i
$\overset{a_1}{\to}$	$(-\alpha_2,\beta_1)$	$(-e_2,ie_1)$	i
$\overset{a_4}{\to}$	$(-\alpha_2,\beta_1)$	$(-e_2,ie_1)$	i
$\overset{a_3}{\to}$	$(-\alpha_2+\beta_1+\beta_2,\beta_1)$	$(ie_1+(-1+i)e_2,ie_1)$	$1+i$
$\overset{a_2}{\to}$	$(\alpha_1+\beta_1-\alpha_2+\beta_2,\alpha_1+\beta_1)$	$((1+i)e_1+(-1+i)e_2,(1+i)e_1)$	2
$\overset{a_1}{\to}$	$(\alpha_1-\alpha_2+\beta_2,\alpha_1)$	$(e_1+(-1+i)e_2,e_1)$	$1-i$
$\overset{a_4}{\to}$	$(\alpha_1+\beta_2,\alpha_1)$	(e_1+ie_2,e_1)	$-i$
$\overset{a_3}{\to}$	$(\alpha_1-\beta_1,\alpha_1-\beta_1-\beta_2)$	$(1-i)e_1,(1-i)e_1-ie_2)$	$-1-i$
$\overset{a_2}{\to}$	$(-\beta_1,-\beta_1-\beta_2)$	$(-ie_1,-ie_1-ie_2)$	-1
$\overset{a_1}{\to}$	$(-\beta_1,-\beta_1-\beta_2)$	$(-ie_1,-ie_1-ie_2)$	-1
$\overset{a_4}{\to}$	$(-\beta_1,-\beta_1-\alpha_2-\beta_2)$	$(-ie_1,-ie_1+(-1-i)e_2)$	$i-1$
$\overset{a_3}{\to}$	$(-\beta_1,-\alpha_2)$	$(-ie_1,-e_2)$	i
$\overset{a_2}{\to}$	$(-\alpha_1-\beta_1,-\alpha_2)$	$((-1-i)e_1,-e_2)$	$1+i$
$\overset{a_1}{\to}$	$(-\alpha_1,-\alpha_2)$	$(-e_1,-e_2)$	1

It follows that $\det_{\mathbb{C}}((\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4)^5\cdot\Lambda_0)$ is homotopic to $[0,1]\ni s\mapsto e^{-4\pi is}$. Thus $\det_{\mathbb{C}}((\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4)^{10}(\widetilde{a}_5\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4)^{-6}\cdot\Lambda_0)$ is homotopic to $[0,1]\ni s\mapsto e^{-2\pi is}$ which in turn implies $\widetilde{q}(\kappa)=-1$.

□

Remark 2.29. The distinguished lift of a relation $\mu:\prod_i t_i=1$ in $\mathrm{Sp}(4,\mathbb{Z})$ is a loop $\psi:S^1\longrightarrow\mathrm{Sp}(4,\mathbb{R})$ and its Maslow-index $\mathrm{ind}_{mas}(\psi)$ is just $\widetilde{q}(\mu)$. (For the definition of the Maslow-index, see [50] p. 45f.).

The lemma implies that $\widetilde{q}_{|_N}:N\longrightarrow\mathbb{Z}$ is an epimorphism. Therefore we get the following theorem

Theorem 2.30. *The central extension*

$$0\longrightarrow\mathbb{Z}\longrightarrow\widetilde{\mathrm{Sp}}(4,\mathbb{Z})\longrightarrow\mathrm{Sp}(4,\mathbb{Z})\longrightarrow 1$$

is given by the element $\widetilde{q} \in \mathrm{Hom}(N, \mathbb{Z})^{\mathrm{Br}_6}$ *that is determined by* $\widetilde{q}(\kappa) = \widetilde{q}(\gamma) = -1,\ \widetilde{q}(\pi) = 0$. *And thus*

$$\widetilde{\mathrm{Sp}}(4, \mathbb{Z}) = \frac{\mathrm{Br}_6 \times \mathbb{Z}}{ker\widetilde{q}}.$$

□

Next we give another description of the distinguished lift. The set of transvections Tr in $\mathrm{Sp}(4, \mathbb{Z})$ forms a conjugacy class. For a transvection $t_v \in \mathrm{Sp}(4, \mathbb{Z})$ and F an arbitrary element of $\mathrm{Sp}(4, \mathbb{Z})$ holds

$$F \circ t_v \circ F^{-1} = t_{F(v)}. \tag{15}$$

Thus for a fixed transvection a

$$\mathrm{Tr} = \{t_v | v \in \mathbb{Z}^4\} = \{t \in \mathrm{Sp}(4, \mathbb{Z}) | \exists x \in \mathrm{Sp}(4, \mathbb{Z}) : t = xax^{-1}\}.$$

Now let $t = x * a = xax^{-1}$ be a transvection. Then x is determined up to transformations that commute with a.

Lemma 2.31. *Fix a transvection* $a \in \mathrm{Sp}(4, \mathbb{Z})$ *and* $\widetilde{a}$ *the distinguished lift of* a. *The distinguished lift* $\widetilde{t}$ *of a transvection* $t = x * a$ *is then*

$$\widetilde{t} = y * \widetilde{a},$$

where y *is an arbitrary lift of* x.

Proof: Let $a = t_v$ and $t = x * a$. Let y be a lift of x. Chose a path $\widetilde{\alpha}$ in $\widetilde{\mathrm{Sp}}(4, \mathbb{R})$ that goes from 1 to y and put $\alpha := \pi \circ \widetilde{\alpha}$. We have to show that $\alpha \cdot \widetilde{t_v} \cdot \alpha^{-}$ is homotopic to $\widetilde{t_{x(v)}}$ in $\mathrm{Sp}(4, \mathbb{R})$.
For all $l \in [0, 1]$ holds $\widetilde{t_{x(v)}}(l) \circ x = x \circ \widetilde{t_v}(l)$, because

$$\begin{aligned} \widetilde{t_{x(v)}}(l)(x(u)) &= x(u) + l\omega(x(v), x(u))x(v) \\ &= x(u + l\omega(v, u)v) \\ &= x(\widetilde{t_v}(l)(u)) \end{aligned}$$

for all $u \in \mathbb{R}^4$.

$$\begin{aligned}
(\alpha \cdot \widetilde{t_v})(s) &= \left\{ \begin{array}{ll} \widetilde{t_v}(2s) & ; s \in [0, \frac{1}{2}] \\ \alpha(2s-1) \circ t_v & ; s \in [\frac{1}{2}, 1] \end{array} \right\} \\
(\widetilde{t_{x(v)}} \cdot \alpha)(s) &= \left\{ \begin{array}{ll} \alpha(2s) & ; s \in [0, \frac{1}{2}] \\ \widetilde{t_{x(v)}}(2s-1) \circ \alpha(1) & ; s \in [\frac{1}{2}, 1] \end{array} \right\} \\
&= \left\{ \begin{array}{ll} \alpha(2s) & ; s \in [0, \frac{1}{2}] \\ \alpha(1) \circ \widetilde{t_v}(2s-1) & ; s \in [\frac{1}{2}, 1] \end{array} \right\}.
\end{aligned}$$

Let

$$h_l(s) = \left\{ \begin{array}{ll} \widetilde{t_v}(3sl) & ; s \in [0, \frac{1}{3}] \\ \alpha(3s-1) \circ \widetilde{t_v}(l) & ; s \in [\frac{1}{3}, \frac{2}{3}] \\ \widetilde{t_{x(v)}}(l + (1-l)(3s-2)) \circ \alpha(1) & ; s \in [\frac{2}{3}, 1] \end{array} \right\}.$$

This defines a homotopy as $h_l(0) = id$ and $h_l(1) = \alpha(1) \circ t_v$. Up to reparametrisation

$$\begin{aligned}
h_0(s) &= \alpha \cdot \widetilde{t_v} \\
h_1(s) &= \widetilde{t_{x(v)}} \cdot \alpha
\end{aligned}$$

and the claim follows.

□

2.4. **The class $\gamma^* \in H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$.** The natural homomorphism $\mathrm{Br}_6 \longrightarrow \mathrm{Map}_2$ restricts to a homomorphism $N \longrightarrow \mathcal{J}_2$. Recall the γ-degree

$$\deg_\gamma : \mathcal{J}_2 \longrightarrow \mathbb{Z}$$

from Definition 2.13. The pullback of $\deg_\gamma$ under the above homomorphism is a Br_6-invariant homomorphism $\gamma^* : N \longrightarrow \mathbb{Z}$. Therefore it represents a class $\gamma^* \in H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$. Clearly

$$\gamma^*(\gamma) = 1 \ , \ \gamma^*(\kappa) = \gamma^*(\pi) = 0.$$

Proposition 2.32. *There is an element $\gamma^* \in \mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}$ such that $\gamma^*(\gamma) = 1$ and $\gamma^*(\kappa) = \gamma^*(\pi) = 0$.*

□

Let

$$0 \longrightarrow \mathbb{Z} \longrightarrow G_{\gamma^*} \longrightarrow \mathrm{Sp}(4,\mathbb{Z}) \longrightarrow 1$$

be the extension defined by γ^*. By definition

$$G_{\gamma^*} = \frac{\mathrm{Br}_6}{\langle \kappa, \pi, [\gamma, *] \rangle}$$

where $[\gamma, *]$ stands for a set of relators that generate commutators of γ with all other elements. Note that this implies that there exists a commutative diagram

$$\begin{tikzcd} \mathrm{J}_2 \arrow[r] \arrow[d, "\deg_\gamma"'] & \mathrm{Map}_2 \arrow[r, "\xi"] \arrow[d, "\rho"] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[d, equal] \\ \mathbb{Z} \arrow[r] & G_{\gamma^*} \arrow[r, "p"] & \mathrm{Sp}(4,\mathbb{Z}) \end{tikzcd} \tag{16}$$

2.5. **Generators and relations in $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$.** We set $\kappa^* := -\widetilde{q} - \gamma^*$. This gives a well defined element of $\mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}$ that satisfies

$$\kappa^*(\kappa) = 1,\ \kappa^*(\pi) = \kappa^*(\gamma) = 0.$$

We prove that the two classes $\kappa^* + \gamma^*, \gamma^*$ generate $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$.

Theorem 2.33. *$H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$ is generated by the two classes $\kappa^* + \gamma^*, \gamma^*$ and isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.*

In order to prove the theorem, we need the following lemma.

Lemma 2.34. *Let $\varphi : N \longrightarrow \mathbb{Z}$ be a Br_6-invariant Homomorphism. Then $\varphi(\pi) = 0$.*

Proof: It suffices to consider φ such that $\varphi(\kappa) = \varphi(\gamma) = 0$. Let $\Sigma_{[1]}$ be a surface of genus two with one boundary component $\partial\Sigma$, Σ_1 the closed surface obtained by capping $\partial\Sigma$ with a disc containing a distinguished point and Σ the same surface without marked point. Consider $\langle \kappa, \pi \rangle_{\mathrm{Br}_6} \subset N$ and set

$$\widehat{\pi_1(\Sigma)} := \frac{\langle \kappa, \pi \rangle_{\mathrm{Br}_6}}{\langle \kappa \rangle_{\mathrm{Br}_6}}.$$

Let $\varphi : \widehat{\pi_1(\Sigma)} \longrightarrow \mathbb{Z}$ be a Br_6-invariant Homomorphism. We have to show that $\varphi \equiv 0$. Recall the commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{2,[1]} \arrow[r, "f_1"] \arrow[dr, "f"] & \mathrm{Map}_{2,1} \arrow[d, "f_2"] \\ & \mathrm{Map}_2 \end{tikzcd} \tag{17}$$

The kernel of the maps f is $\widehat{\pi_1(\Sigma)}$ and from the diagram we see that it fits into an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widehat{\pi_1(\Sigma)} \longrightarrow \pi_1(\Sigma) \longrightarrow 1. \tag{18}$$

The extension (18) is the pull back of the extension $\mathbb{Z} \longrightarrow \mathrm{Map}_{2,[1]} \longrightarrow \mathrm{Map}_{2,1}$ under the inclusion $i : \pi_1(\Sigma) \hookrightarrow \mathrm{Map}_{2,1}$.

$$\begin{tikzcd} & & 1 \arrow[d] & 1 \arrow[d] & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \widehat{\pi_1(\Sigma)} \arrow[r] \arrow[d] & \pi_1(\Sigma) \arrow[r] \arrow[d, hook] & 1 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Map}_{2,[1]} \arrow[r] \arrow[d] & \mathrm{Map}_{2,1} \arrow[r] \arrow[d] & 1 \\ & & \mathrm{Map}_2 \arrow[r, equal] \arrow[d] & \mathrm{Map}_2 \arrow[r] \arrow[d] & 1 \\ & & 1 & 1 & \end{tikzcd}$$

Johnson proves in [35], Section 3, Lemma 3 that $\widehat{\pi_1(\Sigma)}$ is isomorphic to the fundamental group of the unit tangent bundle $S^1\Sigma$ of Σ and that the long exact homotopy sequence associated to the bundle $S^1 \hookrightarrow S^1\Sigma \longrightarrow \Sigma$ gives the extension (18). The fundamental group of Σ has the well known presentation

$$\pi_1(\Sigma) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 | \prod_{i=1}^{2} [\alpha_i, \beta_i] \rangle.$$

Recall from Remark 2.5 that the image of α_i (resp. β_i) under the map $i : \pi_1(\Sigma) \longrightarrow \mathrm{Map}_{2,1}$ is represented by

$$\tau_{\alpha_i+} \cdot \tau_{\alpha_i-}^{-1} \quad \text{resp.} \quad \tau_{\beta_i+} \cdot \tau_{\beta_i-}^{-1}$$

in the notation of Remark 2.5. These Dehn twists have natural lifts to $\mathrm{Map}_{2,[1]}$. In this way we get natural lifts $\widehat{\alpha}_i, \widehat{\beta}_i \in \widehat{\pi_1(\Sigma)}$ of α_i, β_i. The Dehn twist $\tau_{\partial\Sigma} \in \mathrm{Map}_{2,[1]}$ around the boundary component on the other hand generates $\mathbb{Z} \subset \widehat{\pi_1(\Sigma)}$. The lift of the single relation in $\pi_1(\Sigma)$ is therefore

$$\prod_{i=1}^{2}[\widehat{\alpha}_i, \widehat{\beta}_i] = \tau_{\partial\Sigma}^k,$$

for a $k \in \mathbb{Z}$. The class of the extension (18) corresponds under the natural isomorphism $H^2(\pi_1(\Sigma), \mathbb{Z}) = H^2(\Sigma, \mathbb{Z})$ to the Euler class and thus $k = \chi_{\mathrm{top}}(\Sigma) = 2$. As a consequence $\varphi(\tau_{\partial\Sigma}) = 0$ and $\varphi : \widehat{\pi_1(\Sigma)} \longrightarrow \mathbb{Z}$ descends to a $\mathrm{Map}_{2,1}$-invariant homomorphism $\varphi : \pi_1(\Sigma) \longrightarrow \mathbb{Z}$. But $\mathrm{Map}_{2,1} = \mathrm{Aut}^+(\pi_1(\Sigma))$ acts transitively on $\pi_1(\Sigma)$ and therefore φ has to be identically zero.

□

Proof of the Theorem: Lemma 2.34 implies that κ^* and γ^* generate $\mathrm{Hom}(N, \mathbb{Z})^{\mathrm{Br}_6}$. Therefore it establishes the first claim of Theorem 2.33. Recall the degree homomorphism $\deg : \mathrm{Br}_6 \longrightarrow \mathbb{Z}$ from Remark 2.7 and Remark 2.17. The elements κ and γ of Br_6 have degree 10 and 12 respectively. This implies that in $\mathrm{Hom}(N, \mathbb{Z})^{\mathrm{Br}_6}$ holds

$$\deg = 10\kappa^* + 12\gamma^*. \tag{19}$$

Thus

$$\frac{\mathrm{Hom}(N, \mathbb{Z})^{\mathrm{Br}_6}}{\mathbb{Z}\langle\deg\rangle} \simeq \mathbb{Z} \oplus \mathbb{Z}_2$$

and the second claim follows.

□

Remark 2.35. For $i = 0, 1$ let $\mathrm{Map}_{2,[i]} = \frac{\mathrm{Br}_6}{N_{2,[i]}}$, i.e. $N_{2,[1]} = \langle\kappa\rangle_{\mathrm{Br}_6}$ and $N_{2,[0]} = \langle\kappa, \pi\rangle_{\mathrm{Br}_6}$. It is easy to see that in both cases $H_1(\mathrm{Map}_{2,[i]}, \mathbb{Z}) = \mathbb{Z}_{10}$ and thus $H^1(\mathrm{Map}_{2,[i]}, \mathbb{Z}) = 0$. Now the Hochschild-Serre spectral sequence yields

$$H^2(\mathrm{Map}_{2,[i]}, \mathbb{Z}) = \frac{\mathrm{Hom}(N_{2,[i]}, \mathbb{Z})^{\mathrm{Br}_6}}{\mathrm{Hom}(\mathrm{Br}_6, \mathbb{Z})}.$$

Restricted to $N_{2,[1]}$ the homomorphism κ^* is equal to $\frac{\deg}{10}$. Thus we conclude that $H^2(\mathrm{Map}_{2,[1]}, \mathbb{Z}) = \mathbb{Z}_{10}$, with generator κ^*. Note that Lemma 2.34 implies that $H^2(\mathrm{Map}_2, \mathbb{Z}) = H^2(\mathrm{Map}_{2,[1]}, \mathbb{Z})$.
The pullback of the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}}(4, \mathbb{Z}) \longrightarrow \mathrm{Sp}(4, \mathbb{Z}) \longrightarrow 1$$

under the homomorphism

$$\xi : \mathrm{Map}_2 \longrightarrow \mathrm{Sp}(4,\mathbb{Z})$$

is represented by the pullback of $(\kappa^* + \gamma^*)$ under the inclusion $N_{2,[0]} \hookrightarrow N$ and thus corresponds to the generator $\kappa^* \in H^2(\mathrm{Map}_2, \mathbb{Z})$. We denote this extension by $\widetilde{\mathrm{Map}}_2$ and get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \widetilde{\mathrm{Map}}_2 \arrow[r] \arrow[d] & \mathrm{Map}_2 \arrow[r] \arrow[d, "\xi"] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \widetilde{\mathrm{Sp}}(4,\mathbb{Z}) \arrow[r] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[r] & 1.
\end{tikzcd}$$

Note that

$$\widetilde{\mathrm{Map}}_2 = \frac{\mathrm{Br}_6}{\langle [\kappa, *], \pi \rangle}$$

and that $\deg : \mathrm{Br}_6 \longrightarrow \mathbb{Z}$ descends to a homomorphism $\deg : \widetilde{\mathrm{Map}}_2 \longrightarrow \mathbb{Z}$.

2.6. **The central extension $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$.** In this section we describe an extension of $\mathrm{Sp}(4,\mathbb{Z})$ by $\mathbb{Z}^2$. Consider the following element of $\mathrm{Hom}(N, \mathbb{Z}^2)^{\mathrm{Br}_6}$

$$\psi := \begin{pmatrix} \kappa^* \\ \gamma^* \end{pmatrix} : N \longrightarrow \mathbb{Z}^2.$$

It defines a central extension

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \widehat{\mathrm{Sp}}(4,\mathbb{Z}) \xrightarrow{\pi} \mathrm{Sp}(4,\mathbb{Z}) \longrightarrow 1, \tag{20}$$

with $\widehat{\mathrm{Sp}}(4,\mathbb{Z}) = \frac{\mathrm{Br}_6 \times \mathbb{Z}^2}{\psi(N)}$. Note that

$$\widehat{\mathrm{Sp}}(4,\mathbb{Z}) = \frac{Br_6}{\ker \psi} = \frac{Br_6}{\langle \pi, [\kappa, *], [\gamma, *] \rangle},$$

where as before $[x, *]$ denotes a set of relators that generate commutators of x with all other elements. Note also that there is a well defined degree homomorphism $\deg : \widehat{\mathrm{Sp}}(4,\mathbb{Z}) \longrightarrow \mathbb{Z}$ induced by $\deg : \mathrm{Br}_6 \longrightarrow \mathbb{Z}$. The extension (20) has two useful properties. The first property is similar to that of a universal central extension.

Proposition 2.36. *Let E be a central extension of $\mathrm{Sp}(4,\mathbb{Z})$ by $\mathbb{Z}$. Then there exists a unique homomorphism $\theta : \widehat{\mathrm{Sp}}(4,\mathbb{Z}) \longrightarrow E$ such that the following diagram* *commutes*

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \widehat{\mathrm{Sp}}(4,\mathbb{Z}) \arrow[r] \arrow[d, "\theta"] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & E \arrow[r] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[r] & 1. \end{tikzcd}$$

Proof: Let E be defined by $\varphi \in \mathrm{Hom}(N,\mathbb{Z})^{\mathrm{Br}_6}$. Then $\varphi = k\kappa^* + l\gamma^*$, for $k, l \in \mathbb{Z}$. There is a unique $\theta' : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ such that $\theta' \circ \psi = \varphi$. The homomorphism $id \times \theta' : \mathrm{Br}_6 \times \mathbb{Z}^2 \longrightarrow \mathrm{Br}_6 \times \mathbb{Z}$ descends to homomorphism $\theta : \frac{\mathrm{Br}_6 \times \mathbb{Z}^2}{\psi(N)} \longrightarrow \frac{\mathrm{Br}_6 \times \mathbb{Z}}{\varphi(N)}$.

□

The second property is the existence of a distinguished lift for transvections.

Proposition 2.37. *Let $t \in \mathrm{Sp}(4,\mathbb{Z})$ be a transvection. Then there exists a distinguished lift $\widehat{t}$ of t in $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$.*

Proof: Let $t \in \mathrm{Sp}(4,\mathbb{Z})$ be a transvection, we define the distinguished lift $\widehat{t}$ in the following way. Denote by $a_i \in \mathrm{Sp}(4,\mathbb{Z})$ and $\widehat{a}_i \in \widehat{\mathrm{Sp}}(4,\mathbb{Z})$ the images of $a_i \in \mathrm{Br}_6$ under the natural quotient maps. Then there exists a $b \in \mathrm{Sp}(4,\mathbb{Z})$ such that t is conjugate to a_2 by b, i.e. $t = b * a_2$. Set $\widehat{t} := \beta * \widehat{a}_2$ for an arbitrary lift β of b. We will show that this is well defined.
Denote by $\widehat{Tr}$ the conjugacy class of $\widehat{a}_2$ in $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$. Then $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$ and $\mathrm{Sp}(4,\mathbb{Z})$ act by conjugation on $\widehat{Tr}$ and Tr respectively. The map $p := \pi|\widehat{Tr}$ is equivariant with respect to these actions.

$$\begin{tikzcd} \widehat{\mathrm{Sp}}(4,\mathbb{Z}) \arrow[r, bend left] \arrow[d, "\pi"'] & \widehat{Tr} \arrow[d, "p"] \\ \mathrm{Sp}(4,\mathbb{Z}) \arrow[r, bend left] & Tr \end{tikzcd}$$

The transformation b is obviously unique up to an element of the stabiliser I_{a_2} of a_2. We must therefore show that if b lies in the stabiliser of a_2 then every lift β of b lies in the stabiliser $I_{\widehat{a}_2}$ of $\widehat{a}_2$. The extension being central it suffices to show that for each $b \in I_{a_2}$ there exists a lift $\beta \in I_{\widehat{a}_2}$. Thus the proof is complete once we have established the following lemma.

Lemma 2.38. *The map $\pi|I_{\widehat{a}_2} \longrightarrow I_{a_2}$ between the stabilisers is surjective.*

Proof of the Lemma: With respect to the symplectic basis $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ in $\mathbb{Z}^4$

$$a_2 = t_{\alpha_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let F be in the stabiliser I_{a_2}, i.e.

$$F * t_{\alpha_1} = t_{F(\alpha_1)} = t_{\alpha_1}$$

The stabiliser is therefore

$$I_{a_2} = \{F \in \mathrm{Sp}(4, \mathbb{Z}) | F(\alpha_1) = \pm\alpha_1\}.$$

We show that such an F has an inverse image in $I_{\widehat{a}_2}$.
I. *Without loss of generality we can assume that $F(\alpha_1) = \alpha_1$.* For if $F(\alpha_1) = -\alpha_1$, we write $F = F' \cdot (-id)$. An inverse image of $-id \in I_{a_2}$ in $\widehat{\mathrm{Sp}}(4, \mathbb{Z})$ that commutes with $\widehat{a}_2$ is given by $(\widehat{a}_1\widehat{a}_2)^3(\widehat{a}_4\widehat{a}_5)^3$. Thus it suffices to show the claim for F'.
Let therefore F be a symplectic transformation given by a matrix of the form:

$$\begin{pmatrix} 1 & n & * & * \\ 0 & 1 & * & * \\ 0 & k & * & * \\ 0 & l & * & * \end{pmatrix}. \tag{21}$$

II. *Without loss of generality we can assume that $n = 0$.* Otherwise we write $F = F' \cdot a_2^n$. As a_2^n obviously has an inverse image in $I_{\widehat{a}_2}$, it suffices to prove the claim for F'.

Let F be as above with $n = 0$. Now the fact that F is symplectic implies that F is given by a matrix of the following form

$$\begin{pmatrix} 1 & 0 & m & p \\ 0 & 1 & 0 & 0 \\ 0 & k & r_1 & r_2 \\ 0 & l & r_3 & r_4 \end{pmatrix}. \tag{22}$$

III. *Without loss of generality we can assume the submatrix $R := \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}$ in (22) to be the unit matrix E_2.* We achieve this by multiplying F on the right with the symplectic matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s_1 & s_2 \\ 0 & 0 & s_3 & s_4 \end{pmatrix},$$

where the submatrix $S := \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}$ is R^{-1}. This matrix clearly has an inverse image in $I_{\widehat{a}_2}$.

Let therefore F be a matrix of the form

$$\begin{pmatrix} 1 & 0 & -l & k \\ 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & l & 0 & 1 \end{pmatrix}. \tag{23}$$

IV. *Without loss of generality we can assume that F has block diagonal form*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}. \tag{24}$$

We achieve this by multiplying F on the right by the transvection $t_{\alpha_1 - k\alpha_2 - l\beta_2}$. As

$$\alpha_1 \cdot (\alpha_1 - k\alpha_2 - l\beta_2) = 0 \tag{25}$$

this transvection lies in I_{a_2}. We claim that it has an inverse image in $I_{\widehat{a}_2}$. Denote by τ_C the class in Map_2 of a Dehn twist along a simple closed curve C that represents $\alpha_1 - k\alpha_2 - l\beta_2 \in H_1(\Sigma, \mathbb{Z})$. Denote by $a_2^{\mathrm{Map}_2} \in \mathrm{Map}_2$ the image of $a_2 \in \mathrm{Br}_6$. This is the class of a Dehn twist along a simple closed curve that represents α_1. It follows from (25) that τ_C and $a_2^{\mathrm{Map}_2}$ commute. Remark 2.35 implies that there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathrm{Map}}_2 & \longrightarrow & \mathrm{Map}_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \Xi} & & \downarrow{\scriptstyle \xi} & & \\ 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \widehat{\mathrm{Sp}}(4,\mathbb{Z}) & \longrightarrow & \mathrm{Sp}(4,\mathbb{Z}) & \longrightarrow & 1. \end{array} \tag{26}$$

Denote the image of $a_2 \in \mathrm{Br}_6$ in $\widetilde{\mathrm{Map}}_2$ by $\widetilde{a}_2$ and let $\widetilde{\tau}$ be an arbitrary lift of τ_C to $\widetilde{\mathrm{Map}}_2$. As τ_C and $a_2^{\mathrm{Map}_2}$ commute, the commutator $[\widetilde{\tau}, \widetilde{a}_2]$ lies in $\mathbb{Z}\langle\kappa\rangle$. The degree homomorphism $\deg : \mathrm{Br}_6 \longrightarrow \mathbb{Z}$ descends to a well defined homomorphism

$\deg : \widetilde{\mathrm{Map}}_2 \longrightarrow \mathbb{Z}$. Now $\deg_{|\mathbb{Z}\langle\kappa\rangle} = \mathbb{Z} \xrightarrow{12} \mathbb{Z}$, but $\deg([\widetilde{\tau}, \widetilde{a}_2]) = 0$. So we conclude that

$$[\widetilde{\tau}, \widetilde{a}_2] = 1 \in \widetilde{\mathrm{Map}}_2.$$

It follows that $\Xi(\widetilde{\tau})$ is a lift of $t_{\alpha_1 - k\alpha_2 - l\beta_2}$ that lies in $I_{\widehat{a}_2}$.

V. *A symplectic matrix F as in (24) has a lift in $I_{\widehat{a}_2}$.* Applying **II.** once again we can assume that F has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}. \tag{27}$$

Such a matrix clearly has a lift in $\widehat{\mathrm{Sp}}(4, \mathbb{Z})$ that commutes with $\widehat{a}_2$. This completes the proof of the lemma.

□

3. Geometry

3.1. **Evaluation of cohomology classes.** Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface. Let $\Delta = \{p_1, ..., p_d\}$ be the discriminant locus of f and fix a base point $b_0 \in \mathbb{P}^1 \setminus \Delta$. Let $\alpha_1, ...\alpha_d$ be loops with base point b_0 such that α_i goes counterclockwise around p_i and such that

$$\pi_1(\mathbb{P}^1 \setminus \Delta) = \langle \alpha_1, ...\alpha_d | \prod_{i=1}^{d} \alpha_i = 1 \rangle$$

is a presentation of the fundamental group. Now assume that for each p_i the monodromy transformation T_i associated to α_i is unipotent of rank one, i.e.

$$T_i = t_i^{k_i}, \tag{28}$$

where t_i is a simple transvection and $k_i \in \mathbb{N}$.

Definition 3.1. *Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface and discriminant locus $\Delta = \{p_1, ..., p_d\}$. We say that f has* unipotent monodromy of rank one *if the monodromy transformation around each p_i has the form (28). We say that f has* simple monodromy *if the monodromy transformation around each p_i is a simple transvection.*

Definition 3.2. *The monodromy factorisation of $f : X \longrightarrow \mathbb{P}^1$ with respect to the basis $\alpha_1, ..., \alpha_d$ of $\pi_1(\mathbb{P}^1 \setminus \Delta)$ is*

$$\mu : \prod_{i=1}^{d} t_i^{k_i} = 1.$$

The degree of a monodromy factorisation μ is

$$\deg(\mu) = \sum_{i=1}^{d} k_i.$$

According to Proposition 2.37 a simple transvection has a distinguished lift in $\widehat{\mathrm{Sp}}(4, \mathbb{Z})$. Using Proposition 2.36 this gives a distinguished lift in every central extension of $\mathrm{Sp}(4, \mathbb{Z})$.

Definition 3.3. *The distinguished lift of a transvection t in a central extension*

$$0 \longrightarrow \mathbb{Z} \longrightarrow E \longrightarrow \mathrm{Sp}(4, \mathbb{Z}) \longrightarrow 1 \tag{29}$$

is the image of the distinguished lift in $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$ *under the homomorphism of Proposition 2.36. The distinguished lift of a monodromy factorisation* μ *is the product of the distinguished lifts of its factors.*

Remark 3.4. It follows from Lemma 2.31 that for the universal covering $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$ this definition coincides with the earlier one in Definition 2.23.

We want to evaluate cohomology classes in $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$ on monodromy factorisations. The distinguished lift of a monodromy factorisation μ in a central extension (29) lies in the kernel of $E \longrightarrow \mathrm{Sp}(4,\mathbb{Z})$ and is thus a number.

Definition 3.5. *Let* $[E]$ *be in* $H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$ *and* μ *be a monodromy factorisation. The evaluation of* $[E]$ *on* μ *is the number* $[E](\mu) \in \mathbb{Z}$ *that is given by the distinguished lift of* μ *in the central extension* E*.*

Remark 3.6. A monodromy factorisation such that the monodromy transformation around each p_i is a product of two commuting transvections (i.e. as in Lemma 1.24) also has a distinguished lift in central extensions of $\mathrm{Sp}(4,\mathbb{Z})$. Therefore we can evaluate cohomology classes on such monodromy factorisations also.

3.2. **The geometrical interpretation of the generator $\kappa^* + \gamma^*$.** In order to give a geometrical interpretation of the class $\kappa^* + \gamma^* \in H^1(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$, we briefly recall some Hodge theory. Here we mostly follow the treatment given in [27].

Let $H^1_{\mathbb{C}}$ be a complex vector space of dimension $2g$. A Hodge structure of weight one on $H^1_{\mathbb{C}}$ consists in

1) a sub-module $H^1_{\mathbb{Z}}$ of rank $2g$ such that $H^1_{\mathbb{C}} = H^1_{\mathbb{Z}} \otimes \mathbb{C}$.
2) a directsum decomposition $H^1_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ with $H^{1,0} = \overline{H^{0,1}}$, where the bar denotes complex conjugation with respect to the real structure $H^1_{\mathbb{R}} = H^1_{\mathbb{Z}} \otimes \mathbb{R}$.

There is a canonical $\mathbb{R}$-linear isomorphism

$$H^{1,0} \hookrightarrow H^1_{\mathbb{C}} \longrightarrow H^1_{\mathbb{R}}$$

that will be denoted by

$$\Upsilon : H^{1,0} \longrightarrow H^1_{\mathbb{R}}. \tag{30}$$

The Hodge decomposition on a compact Kähler manifold X together with the lattice $H^1(X,\mathbb{Z})$ define a weight one Hodge structure on the first cohomology $H^1(X,\mathbb{C})$.

A polarisation of a weight one Hodge structure is an integral alternating bilinear form Q on the lattice $H^1_{\mathbb{Z}}$ such that the following bilinear relations hold.

i) $Q(u,u')=0$ if $u,u' \in H^{1,0}$ and
ii) $iQ(u,\bar{u})>0$ if $0 \neq u \in H^{1,0}$.

Let (X,ω) be a polarised manifold. Then the polarisation $\omega \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ induces a polarisation Q of the weight one Hodge structure on $H^1(X,\mathbb{C})$,

$$Q(\sigma,\tau) := \int_X \sigma \wedge \tau \wedge \omega^{\dim(X)-1}.$$

For (X,ω) a principally polarised abelian variety $(H^1_{\mathbb{Z}},Q)$ is unimodular.

Let $(H^1_{\mathbb{Z}},Q)$ be a unimodular, symplectic lattice and $(e_1,...,e_g,f_1,...,f_g)$ a basis of $H^1_{\mathbb{Z}}$ such that the symplectic form Q is given by

$$\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

A weight one Hodge structure on $H^1_{\mathbb{C}} := H^1_{\mathbb{Z}} \otimes \mathbb{C}$, polarised by Q is then given by a g-dimensional subspace F^1 of $\mathbb{C}^{2g}$, that satisfies the two bilinear relations i) and ii). Let $(\omega_1,...,\omega_g)$ be a basis of F^1. Then

$$(\omega_1,...,\omega_g) = (e_1,...,e_g,f_1,...,f_g) \cdot \Omega$$

with a complex $2g \times g$-matrix

$$\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$$

called the period matrix with respect to the two bases. The period matrix can be normalised to

$$\Omega = \begin{pmatrix} E_g \\ Z \end{pmatrix}$$

By the first bilinear relation Z is symmetric and by the second its imaginary part $\operatorname{Im} Z$ is positive definite. In this way the classifying space of weight one Hodge structures

$$E = \{F^1 \in G(g,H^1_{\mathbb{C}}) | Q(u,u')=0 \,\forall u,u' \in F^1 \text{ and } iQ(u,\bar{u})>0 \,\forall 0 \neq u \in F^1\},$$

where $G(g,H^1_{\mathbb{C}})$ denotes the Grassmannian of g-dimensional subspaces of $H^1_{\mathbb{C}}$, can be identified with the Siegel upper half-space

$$\mathfrak{h}_g = \{Z \in M(g \times g,\mathbb{C}) | Z^{\mathsf{T}} = Z \text{ and } \operatorname{Im} Z > 0\}.$$

A variation of Hodge structure is the parametrised version of a Hodge structure. A variation of Hodge structure of weight one consists in a complex manifold C, a local system $\mathcal{H}^1_{\mathbb{Z}}$ with coefficients in $\mathbb{Z}^{2g}$ on C, a flat holomorphic connection ∇ in the holomorphic vector bundle $\mathcal{H}^1_{\mathbb{C}} = \mathcal{H}^1_{\mathbb{Z}} \otimes \mathcal{O}_C$ such that the sections of $\mathcal{H}^1_{\mathbb{Z}}$ are ∇-flat and a holomorphic subbundle

$$\mathcal{F}^1 \subset \mathcal{H}^1_{\mathbb{C}}$$

such that there is a $\mathcal{C}^\infty$-decomposition

$$\mathcal{H}^1_{\mathbb{C}} = \mathcal{F}^1 \oplus \overline{\mathcal{F}^1}.$$

A polarisation of a variation of Hodge structure is a symplectic structure on $\mathcal{H}^1_{\mathbb{Z}}$, that polarises the Hodge structure at each point.

Let $(D^*, \mathcal{H}^1_{\mathbb{Z}}, \mathcal{H}^1_{\mathbb{C}}, \nabla, \mathcal{F}^1)$ be a variation of Hodge structure over the punctured disc D^*. The bundle $\mathcal{H}^1_{\mathbb{C}}$ has a canonical extension $\widetilde{\mathcal{H}}^1_{\mathbb{C}}$ to D as a holomorphic vector bundle. We will explain this extension below. Furthermore by the nilpotent orbit theorem the subbundle $\mathcal{F}^1$ extends canonically to a holomorphic subbundle $\widetilde{\mathcal{F}}^1$ of $\widetilde{\mathcal{H}}^1_{\mathbb{C}}$.

Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism that is smooth over $C := \mathbb{P}^1 \setminus \Delta$, where $\Delta = \{p_1, ..., p_d\}$. The Hodge decomposition $H^1(X_b, \mathbb{C}) = H^{1,0}(X_b) \oplus H^{0,1}(X_b)$ of a smooth fibre X_b defines a variation of Hodge structure $(C, \mathcal{H}^1_{\mathbb{Z}}, \mathcal{H}^1_{\mathbb{C}}, \nabla, \mathcal{F}^1)$. Let $X_1 := f^{-1}(C)$ and $f_1 := f_{|X_1} : X_1 \longrightarrow C$ be the smooth part of f. Then

$$\begin{aligned} \mathcal{H}^1_{\mathbb{Z}} &= R^1 f_{1*}\mathbb{Z} \\ \mathcal{H}^1_{\mathbb{C}} &= R^1 f_{1*}\mathbb{Z} \otimes \mathcal{O}_C \\ \mathcal{F}^1 &= f_{1*}\Omega^1_{X_1/C}. \end{aligned}$$

Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface. What we constantly have in mind in what follows is the restriction of a Lagrangian fibration to a general line in $\mathbb{P}^2$. A smooth fibre X_b is then a complex torus $\frac{V_b}{\Lambda_b}$ and V_b and Λ_b are naturally identified with $H^0(X_b, \Omega^1_{X_b})^*$ and $H_1(X_b, \mathbb{Z})$ respectively. The polarisation ω_b of X_b induces via

$$Q_b(\sigma, \tau) = \int_{X_b} \sigma \wedge \tau \wedge \omega_b$$

a polarisation of the Hodge structure. Let $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ be a basis of $H_1(X_b, \mathbb{Z})$ such that ω_b is given by

$$\begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}. \tag{31}$$

Then Q_b with respect to the dual basis (e_1, e_2, f_1, f_2) in $H^1(X_b, \mathbb{Z})$ is also given by the matrix (31). Over C the Hodge structures of the fibres form a polarised variation of Hodge structure. Let

$$\rho : \pi_1(S, b_0) \longrightarrow \mathrm{Sp}(4, \mathbb{Z})$$

be the monodromy representation of the local system $\mathcal{H}^1_{\mathbb{Z}}$ and S_i a monodromy transformation. Then

$$S_i = T_i^{\,-\mathsf{T}},$$

where T_i is the monodromy of $\mathcal{H}_{1,\mathbb{Z}}$ with respect to the dual basis. Using the canonical extension of $\mathcal{H}^1_{\mathbb{C}}$ at each critical value p_i we get a canonical extension $\widetilde{\mathcal{H}^1_{\mathbb{C}}}$ of $\mathcal{H}^1_{\mathbb{C}}$ to $\mathbb{P}^1$ and an extension $\widetilde{\mathcal{F}}^1$ of $\mathcal{F}^1$ as a subbundle of $\widetilde{\mathcal{H}^1_{\mathbb{C}}}$. The bundle $\mathcal{F}^1$ as well as its extension $\widetilde{\mathcal{F}}^1$ will be called the *Hodge bundle* of f. Let $\mu : \pi_1(S, b_0) \longrightarrow \mathrm{Sp}(4, \mathbb{Z})$ be the monodromy representation of the fibration f and assume that the monodromy around each critical value p_i is unipotent of rank one. We claim that the evaluation of the class $\kappa^* + \gamma^*$ on a monodromy factorisation μ of f gives the first Chern class of the Hodge bundle $\widetilde{\mathcal{F}}^1$.

Theorem 3.7. *Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface and unipotent monodromy of rank one. Then*

$$(\kappa^* + \gamma^*)(\mu) = c_1(\widetilde{\mathcal{F}^1}),$$

where $\widetilde{\mathcal{F}^1}$ is the Hodge bundle of f and μ is a monodromy factorisation of f.

Before proving this theorem we will first recall some more facts concerning variations of Hodge structure.

3.2.1. *The canonical extension.* For a polarised variation of Hodge structure $(D^*, \mathcal{H}^1_{\mathbb{Z}}, \mathcal{H}^1_{\mathbb{C}}, \nabla, \mathcal{F}^1, Q)$ over the punctured disc $D^* = \{z \in \mathbb{C} |\, 0 < |z| < 1\}$ the canonical extension of the bundle $\mathcal{H}^1_{\mathbb{C}}$ is defined in the following way. A holomorphic vector bundle $\mathcal{V}$ over the punctured disc is trivial. So each trivialisation of $\mathcal{V}$ defines an extension to the disc D. One singles out a privileged extension using the flat structure. Let $z_0 \in D^*$ a base point and $e_1(z_0), ..., e_{2g}(z_0)$ a basis of $\mathcal{H}^1_{\mathbb{Z},z_0}$. By parallel transport we get a multi-valued, flat frame $(e_1(z), ..., e_{2g}(z))$ over D^*.

Denote by S the monodromy transformation corresponding to the counterclockwise generator of $\pi_1(D^*, z_0)$, assume S to be unipotent and let $M := \log S$. The above frame then satisfies

$$(e_1(\exp(2\pi i)z), ..., e_{2g}(\exp(2\pi i)z)) = (e_1(z), ..., e_{2g}(z)) \cdot S.$$

Setting

$$e'_j(z) := \exp(-\frac{\log(z)}{2\pi i}M)e_j(z)$$

for $j = 1, ..., 2g$ therefore defines a single-valued, holomorphic frame over D^*. This frame defines the canonical extension $\widetilde{\mathcal{H}^1_{\mathbb{C}}}$.

3.2.2. *The period map and the nilpotent orbit theorem.* The pullback of the local system $\mathcal{H}^1_{\mathbb{Z}} \longrightarrow D^*$ under the universal covering map

$$\begin{array}{rcl} \exp : \mathfrak{h}_1 & \longrightarrow & D^* \\ w & \mapsto & \exp(2\pi i w) \end{array}$$

is a constant sheaf and an isomorphism of constant sheaves

$$\exp^* \mathcal{H}^1_{\mathbb{Z}} = \mathfrak{h}_1 \times H^1_{\mathbb{Z}}$$

is fixed by stipulating $\mathcal{H}^1_{\mathbb{Z},z_0} = H^1_{\mathbb{Z}}$. This induces a trivialisation of the bundle $\exp^* \mathcal{H}^1_{\mathbb{C}}$,

$$\exp^* \mathcal{H}^1_{\mathbb{C}} = \mathfrak{h}_1 \times H^1_{\mathbb{C}},$$

where $H^1_{\mathbb{C}} = H^1_{\mathbb{Z}} \otimes \mathbb{C}$. Pulling back the subspaces $\mathcal{F}^1_z \subset \mathcal{H}^1_{\mathbb{C},z}$ yields holomorphically varying subspaces F^1_w of the fixed vector space $H^1_{\mathbb{C}}$. They satisfy

$$F^1_{w+1} = S^{-1} \cdot F^1_w.$$

To each F^1_w corresponds a normalised period matrix

$$\Omega_w = \begin{pmatrix} E_g \\ Z_w \end{pmatrix}$$

such that $F^1_w = \mathrm{span}_{\mathbb{C}}(\Omega_w)$. It follows that

$$\Omega_{w+1} = S^{-1} \cdot \Omega_w. \tag{32}$$

The map

$$\begin{array}{rccc} \widetilde{\phi}: & \mathfrak{h}_1 & \longrightarrow & \mathfrak{h}_g \\ & w & \mapsto & Z_w \end{array}$$

induces the period map

$$\phi : D^* \longrightarrow \mathfrak{h}_g / \operatorname{Sp}(2g, \mathbb{Z}).$$

but because of (32) it does not descend to a map $D^* \longrightarrow \mathfrak{h}_g$. Setting

$$\widetilde{\psi}(w) := \exp(wM)\widetilde{\phi}(w)$$

however, we get a map from the upper half-plane into the compact dual of E, i.e. into the space of g-planes that satisfy the first bilinear relation

$$\check{E} = \{F^1 \in G(g, H^1_{\mathbb{C}}) | Q(u, u') = 0 \, \forall u, u' \in F^1\}.$$

The map $\tilde{\psi} : \mathfrak{h}_1 \longrightarrow \check{E}$ satisfies $\tilde{\psi}(w+1) = \tilde{\psi}(w)$ and thus descends to a map

$$\psi : D^* \longrightarrow \check{E},$$

where $\psi(z) := \widetilde{\psi}(\frac{\log z}{2\pi i})$. By the nilpotent orbit theorem the map ψ has a removable singularity at the origin, see [27], p. 79. It defines a single-valued family of holomorphically varying subspaces of $H^1_{\mathbb{C}}$,

$$\widetilde{F}^1_z := \operatorname{span}_{\mathbb{C}}(\exp(\frac{\log z}{2\pi i} M)\Omega_{\frac{\log z}{2\pi i}})$$

The limiting Hodge filtration is defined as $\widetilde{F}^1_0 := \psi(0) \in \check{E}$. Let $(e_1, ..., e_g, f_1, ..., f_g)$ be a multi-valued flat Q-symplectic frame in $\mathcal{H}^1_{\mathbb{Z}}$. As above we define the single-valued, holomorphic, Q-symplectic frame $(e'_1, ..., e'_g, f'_1, ..., f'_g)$. The corresponding trivialisation

$$\widetilde{\mathcal{H}}^1_{\mathbb{C}} \overset{\theta}{\simeq} D \times H^1_{\mathbb{C}}$$

is the one that defines the canonical extension. Under the isomorphism θ the space $\mathcal{F}^1_z \subset \mathcal{H}^1_{\mathbb{C},z}$ corresponds to $\widetilde{F}^1_z \subset H^1_{\mathbb{C}}$. And the canonical extension of $\mathcal{F}^1$ is defined by

$$\widetilde{\mathcal{F}}^1_0 = \theta^{-1}(\widetilde{F}^1_0). \tag{33}$$

Define sections of $\widetilde{\mathcal{H}}^1_{\mathbb{C}}$ by

$$(\omega_1, ..., \omega_g)_z := (e'_1, ..., e'_g, f'_1, ..., f'_g)_z \cdot \exp(\frac{\log z}{2\pi i} N)\Omega_{\frac{\log z}{2\pi i}}.$$

These sections extend over D, as both $(e'_1, ..., e'_g, f'_1, ..., f'_g)_z$ and $\exp(\frac{\log z}{2\pi i} N)\Omega_{\frac{\log z}{2\pi i}}$ extend over D. Over D^* they obviously form a holomorphic frame in $\mathcal{F}^1$ and because of (33) the extended sections form a holomorphic frame in $\widetilde{\mathcal{F}}^1$.

Let $f : X \longrightarrow \mathbb{P}^1$ be a fibration as in Theorem 3.7 with $\Delta = \{p_1, ..., p_d\}$. Pick one

p_i and let $U_i' \simeq D$ be a small disc centered at p_i. Let S be the monodromy transformation of $\mathcal{H}^1_{\mathbb{Z}}$ corresponding to the counterclockwise generator of $\pi_1(D^*, z_0)$. By assumption $S = s^k$ for a simple transvection s and $k \in \mathbb{N}$. Let

$$(e_1, e_2, f_1, f_2)_{\frac{\log z}{2\pi i}},\ z \in D^*$$

be a multi-valued frame in $\mathcal{H}^1_{\mathbb{Z}}$ such that Q with respect to this frame is given by the matrix

$$\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$$

and such that

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -k & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The logarithm is then

$$M = \log S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The normalised period matrix with respect to $(e_1, e_2, f_1, f_2)_{\frac{\log z}{2\pi i}}$ will be denoted by

$$\Omega_{\frac{\log z}{2\pi i}} = \begin{pmatrix} E_2 \\ Z_{\frac{\log z}{2\pi i}} \end{pmatrix}$$

where

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}.$$

As $\Omega_{w+1} = S^{-1} \cdot \Omega_w$, the entry z_{11} is a multi-valued, holomorphic function in $z \in D^*$, whereas the other entries of Z are single-valued. The matrix

$$\exp(\frac{\log z}{2\pi i} M) \cdot \Omega_{\frac{\log z}{2\pi i}} = \begin{pmatrix} E_2 \\ W_{\frac{\log z}{2\pi i}} \end{pmatrix}$$

on the other hand is single-valued and by the nilpotent orbit theorem extends over 0. The entries of

$$W = \begin{pmatrix} w_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$$

are thus single-valued holomorphic functions on D and

$$z_{11} = w_{11} + k\frac{\log z}{2\pi i}.$$

The frame

$$(e_1', e_2', f_1', f_2') := (e_1, e_2, f_1, f_2) \cdot \exp(-\frac{\log z}{2\pi i} M)$$

in $\mathcal{H}^1_{\mathbb{C}}$ is single-valued and holomorphic.

The canonical $\mathbb{R}$-linear isomorphism

$$\mathcal{F}^1 \xrightarrow{\Upsilon} \mathcal{H}^1_{\mathbb{R}}$$

endows the bundle $\widetilde{\mathcal{F}^1} \longrightarrow D$ over the punctured disc D^* with a flat connection ∇, an integral structure and an integral symplectic structure $\Upsilon^* Q$. We will use this to establish the relationship between the monodromy and the first Chern class.

Lemma 3.8. *The symplectic structure $\Upsilon^* Q$ in $\mathcal{F}^1 \longrightarrow D^*$ can be deformed to a symplectic structure Q' that extends to a symplectic structure Q' in $\widetilde{\mathcal{F}}^1 \longrightarrow D$.*

Proof: We construct Q' by modifying $\Upsilon^* Q$. Let (ω_1, ω_2) be the holomorphic frame in $\widetilde{\mathcal{F}}^1$ defined by

$$(\omega_1, \omega_2)_z := (e_1', e_2', f_1', f_2')_z \cdot \begin{pmatrix} E_2 \\ W_{\frac{\log z}{2\pi i}} \end{pmatrix}, \textit{ i.e.}$$

$$\begin{aligned} \omega_1 &= e_1' + w_{11} f_1' + z_{12} f_2' \\ \omega_2 &= e_2' + z_{12} f_1' + z_{22} f_2'. \end{aligned}$$

The sections (e_1', e_2', f_1', f_2') are not real valued. We consider therefore their real parts

$$\begin{aligned} \mathrm{Re}(e_1') &= e_1 + k\frac{\arg z}{2\pi} f_1 =: g_1 \\ \mathrm{Re}(e_2') &= e_2 \\ \mathrm{Re}(f_1') &= f_1 \\ \mathrm{Re}(f_2') &= f_2, \end{aligned}$$

which form a frame in $\mathcal{H}^1_{\mathbb{R}}$ over D^*, that is symplectic with respect to Q. In the two (real) bases $(\omega_1, \omega_2, i\omega_1, i\omega_2)$ and (g_1, e_2, f_1, f_2) of $\mathcal{F}^1$ and $\mathcal{H}^1_{\mathbb{R}}$ respectively, the isomorphism $\mathcal{F}^1 \xrightarrow{\Upsilon} \mathcal{H}^1_{\mathbb{R}}$ is given by

$$A_z = \begin{pmatrix} E_2 & 0 \\ \operatorname{Re} W & -\operatorname{Im} Z \end{pmatrix}, \ z \in D^*.$$

Then

$$A_z^{-1} = \begin{pmatrix} E_2 & 0 \\ (\operatorname{Im} Z)^{-1} \operatorname{Re} W & -(\operatorname{Im} Z)^{-1} \end{pmatrix}$$

and the columns of A_z^{-1} define a frame $(\sigma_1, ..., \sigma_4)$ in $\mathcal{F}^1$ that is by construction symplectic for $\Upsilon^* Q$.

$$A_z^{-1} \ = \ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\operatorname{Im} z_{22} \operatorname{Re} w_{11} - \operatorname{Im} z_{12} \operatorname{Re} z_{12}}{d(z)} & \frac{\operatorname{Im} z_{22} \operatorname{Re} z_{12} - \operatorname{Im} z_{12} \operatorname{Re} w_{22}}{d(z)} & \frac{-\operatorname{Im} z_{22}}{d(z)} & \frac{\operatorname{Im} z_{12}}{d(z)} \\ \frac{\operatorname{Im} z_{11} \operatorname{Re} z_{12} - \operatorname{Im} z_{12} \operatorname{Re} w_{11}}{d(z)} & \frac{\operatorname{Im} z_{11} \operatorname{Re} w_{22} - \operatorname{Im} z_{12} \operatorname{Re} z_{12}}{d(z)} & \frac{\operatorname{Im} z_{12}}{d(z)} & \frac{-\operatorname{Im} z_{11}}{d(z)} \end{pmatrix}$$

where

$$d(z) := \det(\operatorname{Im} Z).$$

Note that the latter is a positive single-valued function on D^*. We write $z = re^{\arg z}$ for $z \in D$. As

$$\operatorname{Im} z_{11} \ = \ -k\frac{\log r}{2\pi} + \operatorname{Im} w_{11}$$

$$\begin{aligned} d(z) \ &= \ \operatorname{Im} z_{11} \operatorname{Im} z_{22} - (\operatorname{Im} z_{12})^2 \\ &= \ -\frac{k_i \operatorname{Im} z_{22}}{2\pi} \log r + \operatorname{Im} w_{11} \operatorname{Im} z_{22} - (\operatorname{Im} z_{12})^2. \end{aligned}$$

The functions w_{11}, z_{12}, z_{22} are single-valued, holomorphic functions on D^* that extend to holomorphic functions on the hole disc D. Furthermore the nilpotent orbit theorem implies that $\operatorname{Im} z_{22}(0) > 0$, see [28], Proposition 13.3. Thus

$$d(z) \longrightarrow \infty, \ \text{as}\, z \longrightarrow 0.$$

This in turn implies that

$$\lim_{z\longrightarrow 0}(\sigma_1,...,\sigma_4)_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\operatorname{Re} z_{12}(0)}{\operatorname{Im} z_{22}(0)} & \frac{\operatorname{Re} z_{22}(0)}{\operatorname{Im} z_{22}(0)} & 0 & -\frac{1}{\operatorname{Im} z_{22}(0)} \end{pmatrix}.$$

The sections $\sigma_1,...,\sigma_4$ therefore extend over D, but $(\sigma_1(0),...,\sigma_4(0))$ is no longer a basis. Setting

$$(\varsigma_1,...,\varsigma_4) := (\sigma_1,\sigma_2,d(z)\sigma_3,\sigma_4)$$

however defines sections that extend to give a frame in $\widetilde{\mathcal{F}}^1$ over the hole disc D. We define a new symplectic structure Q' in $\widetilde{\mathcal{F}}^1 \longrightarrow D$ by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

with respect to the frame $(\varsigma_1,...,\varsigma_4)$. The symplectic structure Υ^*Q in $\widetilde{\mathcal{F}}^1_{|D^*}$ with respect to this frame is given by

$$\begin{pmatrix} 0 & 0 & d(z) & 0 \\ 0 & 0 & 0 & 1 \\ -d(z) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

As d is a positive $\mathcal{C}^\infty$-functions on D^* $Q'_{|D^*}$ is a deformation of Υ^*Q.

□

Lemma 3.9. *The symplectic structure Q' constructed in Lemma 3.8 is such that the complex structure I of $\widetilde{\mathcal{F}}^1 \longrightarrow D$ tames $-Q'$.*

Proof: We have to show that

$$-Q'(\cdot, I\cdot) > 0.$$

We use the same notation as in the proof of Lemma 3.8. In the basis $(\omega_1, \omega_2, i\omega_1, i\omega_2)$ $-Q'$ is given by

$$A_z^{\mathsf{T}} \begin{pmatrix} 0 & 0 & -\frac{1}{d} & 0 \\ 0 & 0 & 0 & -1 \\ \frac{1}{d} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} A_z$$

$$= \begin{pmatrix} 0 & \operatorname{Re} z_{12} - \frac{\operatorname{Re} z_{12}}{d} & \frac{\operatorname{Im} z_{11}}{d} & \frac{\operatorname{Im} z_{12}}{d} \\ \frac{\operatorname{Re} z_{12}}{d} - \operatorname{Re} z_{12} & 0 & \operatorname{Im} z_{12} & \operatorname{Im} z_{22} \\ -\frac{\operatorname{Im} z_{11}}{d} & -\operatorname{Im} z_{12} & 0 & 0 \\ -\frac{\operatorname{Im} z_{12}}{d} & -\operatorname{Im} z_{22} & 0 & 0 \end{pmatrix}.$$

Therefore $-Q'(\cdot, I\cdot)$ is represented by

$$\begin{pmatrix} \frac{\operatorname{Im} z_{11}}{d} & \frac{\operatorname{Im} z_{12}}{d} & 0 & \operatorname{Re} z_{12}(\frac{1}{d} - 1) \\ \operatorname{Im} z_{12} & \operatorname{Im} z_{22} & \operatorname{Re} z_{12}(1 - \frac{1}{d}) & 0 \\ 0 & 0 & \frac{\operatorname{Im} z_{11}}{d} & \operatorname{Im} z_{12} \\ 0 & 0 & \frac{\operatorname{Im} z_{12}}{d} & \operatorname{Im} z_{22} \end{pmatrix}$$

for $z \in D^*$. This matrix is positive definite as all its principal minors have positive determinant. In the limit as $z \longrightarrow 0 \quad -Q'(\cdot, I\cdot)$ is represented by

$$\begin{pmatrix} \frac{1}{\operatorname{Im} z_{22}(0)} & 0 & 0 & -\operatorname{Re} z_{12}(0) \\ \operatorname{Im} z_{12}(0) & \operatorname{Im} z_{22}(0) & \operatorname{Re} z_{12}(0) & 0 \\ 0 & 0 & \frac{1}{\operatorname{Im} z_{22}(0)} & \operatorname{Im} z_{12}(0) \\ 0 & 0 & 0 & \operatorname{Im} z_{22}(0) \end{pmatrix}$$

and this is likewise positive definite.

$\square$

Let $\widetilde{\mathcal{F}}^1 \longrightarrow \mathbb{P}^1$ be the canonical extension of $\mathcal{F}^1 \longrightarrow \mathbb{P}^1 \setminus \Delta$. We use the construction of Lemma 3.8 to construct a symplectic structure $\widetilde{Q}$ in $\widetilde{\mathcal{F}}^1$.

Using the canonical isomorphism Υ we identify $\mathcal{F}^1$ with $\mathcal{H}^1_{\mathbb{R}}$ over $\mathbb{P}^1 \setminus \Delta$. In this way $\mathcal{F}^1$ becomes endowed with a flat connection, an integral structure $\mathcal{H}^1_{\mathbb{Z}}$ and an integral symplectic structure Q. Let $\Delta = \{p_1, ..., p_d\}$. We choose small discs $U_i \supset U_i' \ni p_i$ centred in p_i and a point $p_0 \in \mathbb{P}^1 \setminus \bigcup_i U_i$. We connect p_0 by a straight line segment $\overline{p_0 p_i}$ to each p_i. The line segments form a star-shaped graph Γ. Now we choose a small neighbourhood $\tau(\Gamma)$ of Γ and put

$$D_0 := \tau(\Gamma) \cup \bigcup_i U_i,$$

as in Figure 5. On the other hand we denote by D_1 be a small neighbourhood of the complement $\mathbb{P}^1 \setminus D_0$.

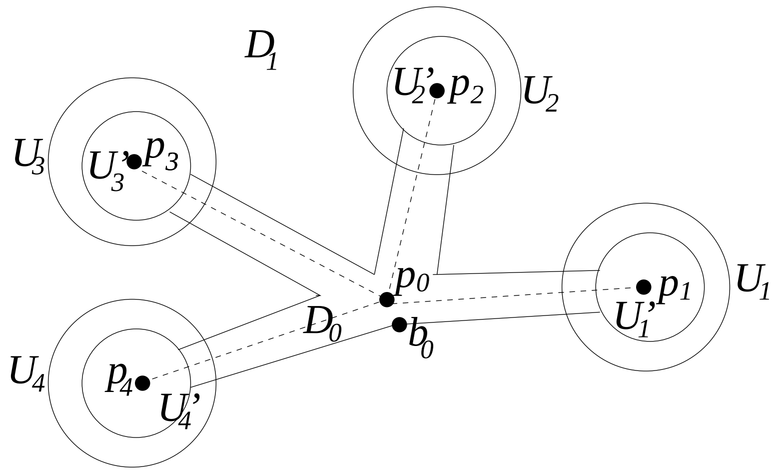

FIGURE 5

Let b_0 be a base point close to p_0 that lies in the overlap of D_0 and D_1 and fix an integral Q-symplectic frame $(e_{1b_0}, e_{2b_0}, f_{1b_0}, f_{2b_0})$ in $\mathcal{F}^1_{b_0}$. Parallel translation then gives a multi-valued, flat, Q-symplectic frame (e_1, e_2, f_1, f_2) in $\mathcal{F}^1$. Now we change the symplectic structure on $U_i \setminus \{p_i\}$. Choose a coordinate $z = x + iy$ centered in p_i such that the situation is as in Figure 6.

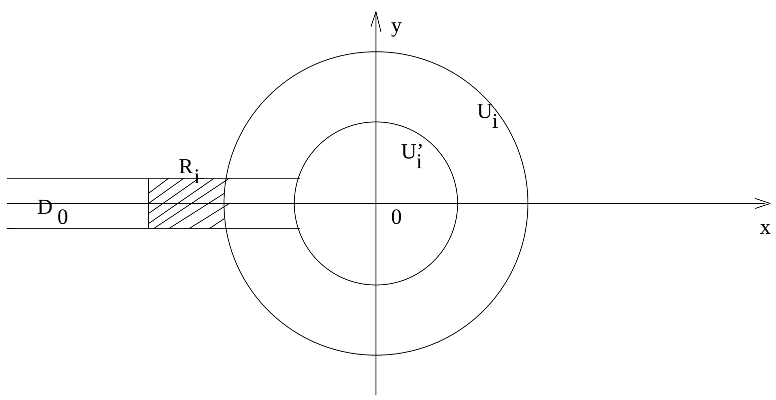

FIGURE 6

Let S_i be the monodromy transformation around p_i and denote its logarithm by M_i. As S_i is unipotent of rank one,

$$S_i = B_i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -k_i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B_i^{-1},$$

for a $B_i \in \mathrm{Sp}(4, \mathbb{Z})$. Then

$$(e_1, e_2, f_1, f_2) \cdot B_i$$

defines a multi-valued, flat, Q-symplectic frame over $\overline{U}_i$. To this frame we apply the construction of the Lemma 3.8. Namely we first construct a single-valued frame

$$\begin{aligned} (e_1', e_2', f_1', f_2') &:= (e_1, e_2, f_1, f_2) \cdot \mathrm{Re}(\exp(-\frac{\log z}{2\pi i} M_i)) \\ &= (e_1, e_2, f_1, f_2) \cdot B_i \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_i \frac{\arg z}{2\pi} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B_i^{-1} \end{aligned}$$

and then modify this to a frame

$$(\varsigma_1^i, ..., \varsigma_4^i) := (e_1', e_2', f_1', f_2') \cdot L_i$$

that extends over p_i. Here L_i is the matrix

$$L_i = B_i \cdot \mathrm{diag}(1, 1, \varphi, 1) \cdot B_i^{-1}$$

with φ a positive $\mathcal{C}^\infty$-function on $\overline{U_i}$ such that

$$\begin{aligned} \varphi &= d(z) \ \mathit{on}\ U_i' \setminus \{p_i\} \ \text{and} \\ \varphi &= 1 \ \mathit{on\ a\ neighbourhood\ of}\ \partial U_i. \end{aligned}$$

The proof of Lemma 3.8 shows that this frame indeed extends over p_i. Requiring that this frame be symplectic defines a symplectic structure $\widetilde{Q}$ over $\overline{U_i}$ that coincides with Q' over U_i' and with Q on ∂U_i. Setting $\widetilde{Q} = Q$ over $\mathbb{P}^1 \setminus \bigcup_i U_i$ we obtain a symplectic structure over $\mathbb{P}^1$.

Lemma 3.10. *Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface and unipotent monodromy of rank one. Then*

$$c_1(\widetilde{\mathcal{F}}^1, \widetilde{Q}) = -(\kappa^* + \gamma^*)(\mu),$$

where $\widetilde{\mathcal{F}}^1$ is the Hodge bundle, $\widetilde{Q}$ the symplectic structure just defined and μ a monodromy factorisation of f.

Proof: We calculate the first Chern class of the symplectic vectorbundle $(\widetilde{\mathcal{F}}^1, \widetilde{Q})$ by calculating the Maslow-index of a transition function.

We can choose D_1 in such a way that $\widetilde{Q}$ coincides with Q over D_1. As D_1 is simply connected parallel translation of $(e_{1b_0}, e_{2b_0}, f_{1b_0}, f_{2b_0})$ over D_1 defines a flat $\widetilde{Q}$-symplectic frame $(\overline{e}_1, \overline{e}_2, \overline{f}_1, \overline{f}_2)$ over D_1. This frame defines a symplectic trivialisation

$$\begin{tikzcd} (\widetilde{\mathcal{F}}^1|_{D_1}, \widetilde{Q}) \arrow[r, "\simeq", "\phi_1"'] \arrow[d] & D_1 \times (\mathbb{R}^4, \omega_0) \arrow[dl] \\ D_1 & \end{tikzcd}$$

of $(\widetilde{\mathcal{F}}^1, \widetilde{Q})$ over D_1 that is flat, i.e. constant sections are flat.
Over D_0 we construct a symplectic trivialisation

$$\begin{tikzcd} (\widetilde{\mathcal{F}}^1|_{D_0}, \widetilde{Q}) \arrow[r, "\simeq", "\phi_0"'] \arrow[d] & D_0 \times (\mathbb{R}^4, \omega_0) \arrow[dl] \\ D_0, & \end{tikzcd}$$

in the following way. Parallel translation of $(e_{1b_0}, e_{2b_0}, f_{1b_0}, f_{2b_0})$ over $D_0 \setminus \bigcup_i U_i$ yields a flat $\widetilde{Q}$-symplectic frame $(\hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2)$ over $D_0 \setminus \bigcup_i U_i$ that extends to the multi-valued flat frame (e_1, e_2, f_1, f_2) on $\bigcup_i \overline{U_i}$. Let z be a coordinate on U_i as above and M_i denote the logarithm of S_i. Then we define on $\overline{U_i}$ the single valued frame

$$\begin{aligned} (e_1', e_2', f_1', f_2') &:= (e_1, e_2, f_1, f_2) \cdot \mathrm{Re}(\exp(-\frac{\log z}{2\pi i} M_i)) \\ &= (e_1, e_2, f_1, f_2) \cdot B_i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_i \frac{\arg z}{2\pi} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B_i^{-1}. \end{aligned}$$

We can modify $(\hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2)$ on the darkend region R_i in Figure 6 such that it stays $\widetilde{Q}$-symplectic and agrees with (e'_1, e'_2, f'_1, f'_2) on $R_i \cap \partial U_i$. The resulting frame on $D_0 \setminus \bigcup_i U_i$ will still be denoted $(\hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2)$. The frame

$$(\varsigma^i_1, ..., \varsigma^i_4) \quad = \quad (e'_1, e'_2, f'_1, f'_2) \cdot L_i$$

is by definition $\widetilde{Q}$-symplectic on $\overline{U_i}$. And over a neighbourhood of ∂U_i in $\overline{U_i}$ it coincides with (e'_1, e'_2, f'_1, f'_2). Therefore it is a $\widetilde{Q}$-symplectic extension of $(\hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2)$ over $\overline{U_i}$. Using the same construction for all i, defines a $\widetilde{Q}$-symplectic extension of $(\hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2)$ over D_0. This defines the trivialisation ϕ_0.

Let

$$\psi = \phi_0 \circ \phi_1^{-1} : \partial D_0 \longrightarrow \mathrm{Sp}(4, \mathbb{R})$$

be the transition function of the two trivialisations along the boundary of D_0. Then the loop ψ in $\mathrm{Sp}(4, \mathbb{R})$ is the expression of the flat symplectic frame $(\overline{e}_1, \overline{e}_2, \overline{f}_1, \overline{f}_2)$ along ∂D_0 in the trivialisation ϕ_0. Thus it represents the monodromy of ∇ along ∂D_0. More specifically it is homotopic to

$$\left(\prod_{i=1}^{d} \tilde{S}_i \right),$$

where $\tilde{S}_i$ is the distinguished lift of S_i in $\widetilde{\mathrm{Sp}}(4, \mathbb{Z})$. This is seen as follows. Along ∂U_i the frame that defines ϕ_0 is

$$(e'_1, e'_2, f'_1, f'_2) \quad = \quad (e_1, e_2, f_1, f_2) \cdot B_i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_i \frac{\arg z}{2\pi} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B_i^{-1}$$

Consequently the flat frame $(\overline{e}_1, \overline{e}_2, \overline{f}_1, \overline{f}_2)$ that defines ϕ_1 is in the above frame given by

$$B_i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -k_i \frac{\arg z}{2\pi} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B_i^{-1},$$

which is just the distinguished lift $\widetilde{S}_i$ of S_i in $\widetilde{\mathrm{Sp}}(4,\mathbb{Z})$. This implies that ψ is homotopic to $\widetilde{S}_d \cdot \ldots \cdot \widetilde{S}_1$ and thus

$$\begin{aligned} c_1(\widetilde{\mathcal{F}}^1, \widetilde{Q}) &= \mathrm{ind}_{mas}(\psi) \\ &= \mathrm{ind}_{mas}(\Big(\widetilde{S}_d \cdot \ldots \cdot \widetilde{S}_1\Big)) \end{aligned}$$

The monodromy transformations of the local system $\mathcal{H}_{1,\mathbb{Z}}$ with respect to the dual basis are $T_i = {S_i}^{-\mathsf{T}}$. The change $S_i \mapsto {S_i}^{-\mathsf{T}}$ is realised by a symplectic base change and thus

$$\mathrm{ind}_{mas}(\Big(\widetilde{S}_d \cdot \ldots \cdot \widetilde{S}_1\Big)) = \mathrm{ind}_{mas}(\Big(\widetilde{T}_d \cdot \ldots \cdot \widetilde{T}_1\Big)).$$

So

$$c_1(\widetilde{\mathcal{F}}^1, \widetilde{Q}) = -(\kappa^* + \gamma^*)(\mu),$$

by Remark 2.29.

$\square$

To prove Theorem 3.7 it remains to prove the following lemma.

Lemma 3.11. *In the situation of Lemma 3.10*

$$c_1(\widetilde{\mathcal{F}}^1, I) = -c_1(\widetilde{\mathcal{F}}^1, \widetilde{Q}),$$

where I denotes the complex structure of $\widetilde{\mathcal{F}}^1$.

Proof: By Lemma 3.9 the complex structure I tames the symplectic structure $-\widetilde{Q}$. Choose a symplectic structure Q_c on $\widetilde{\mathcal{F}}^1$ that is compatible with the complex structure I and consider the family of alternating bilinear forms in $\widetilde{\mathcal{F}}^1$

$$Q_t := (1-t)Q_c - t\widetilde{Q},\ t \in [0,1].$$

This is a family of symplectic structures with $Q_0 = Q_c$ and $Q_1 = -\widetilde{Q}$. For as I tames both $-\widetilde{Q}$ and Q_c, it also tames Q_t. Therefore Q_t is in particular non-degenerate. Now the claim follows as

$$c_1(\widetilde{\mathcal{F}}^1, I) = c_1(\widetilde{\mathcal{F}}^1, Q_c) = c_1(\widetilde{\mathcal{F}}^1, -\widetilde{Q}) = -c_1(\widetilde{\mathcal{F}}^1, \widetilde{Q}).$$

$\square$

Proof of the Theorem: The Lemmata 3.10 and 3.11 now imply Theorem 3.7.

$\square$

Corollary 3.12. *Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface and unipotent monodromy of rank one. Then*

$$\deg(\mu) = 10c_1(\widetilde{\mathcal{F}^1}) + 2\gamma^*(\mu),$$

where μ is a monodromy factorisation of f.

Proof: Recall equation (19)

$$\deg(\mu) = (10\kappa^* + 12\gamma^*)(\mu).$$

Thus

$$\deg(\mu) = 10(\kappa^* + \gamma^*)(\mu) + 2\gamma^*(\mu)$$

and the claim follows from theorem 3.7

□

Remark 3.13. Theorem 3.7 and Corollary 3.12 remain valid in case the monodromy around each point p_i is a product of two commuting transvections (i.e. as in Lemma 1.24). The proofs are strictly analogue.

Consider now a Lagrangian fibration $f : X \longrightarrow \mathbb{P}^2$ with principally polarised fibres. We say that f has unipotent monodromy of rank one (simple monodromy) if the restriction $f_{X_l} : X_l \longrightarrow l$ of f to a general line l in $\mathbb{P}^2$ has unipotent monodromy of rank one (simple monodromy). For the following theorem we assume the fibration to be projective, as we invoke a theorem of Matsushita [45] that requires this. We do not however assume that the family of fibrewise principal polarisations ω is induced by a global polarisation.

Furthermore: *From now on we assume that the general singular fibre of $f : X \longrightarrow \mathbb{P}^2$ has a reduced component.*

Theorem 3.14. *Let $f : X \longrightarrow \mathbb{P}^2$ be a projective Lagrangian fibration with principally polarised fibres and unipotent monodromy of rank one. Let $l \subset \mathbb{P}^2$ be a general line. Then*

$$(\kappa^* + \gamma^*)(\mu_l) = 3,$$

where μ_l denotes a monodromy factorisation of f_{X_l}.

Proof: Let l be a general line in $\mathbb{P}^2$. Matsushita proves in [45] Theorem 1.2 for a projective Lagrangian fibration $f : X \longrightarrow S$

$$R^1 f_* \mathcal{O}_X \simeq \Omega^1_S.$$

Our fibration $f : X \longrightarrow \mathbb{P}^2$ satisfies locally around a smooth point of Δ the assumptions of Proposition 2.1 in [45], as the monodromy is unipotent of rank one and the general singular fibre has a reduced component. This implies that locally around a smooth point of Δ we are in case (1) of Theorem 4.1. in [45] with $G_1 = G_2 = \{1\}$, i.e. locally around a smooth point of Δ the fibration is a toroidal model of type II in the sense of [45] Definition 2.9. In particular the singular fibres are reduced with normal crossings. Consider the restriction $f_{X_l} : X_l \longrightarrow l$ and let $D := f^{-1}_{X_l}(l \cap \Delta)$. By a result of Steenbrink [27] p.130 the canonical extension of the Hodge bundle of $f_{X_{l \setminus \Delta}} : X_{l \setminus \Delta} \longrightarrow (l \setminus \Delta)$ is $\widetilde{\mathcal{F}}^1 = f_* \Omega^1_{X_l/l}(\log(D))$ and the quotient $\widetilde{\mathcal{H}}^1 / \widetilde{\mathcal{F}}^1$, where $\widetilde{\mathcal{H}}^1$ is the canonical extension of $\mathcal{H}^1$, is $R^1 f_* \mathcal{O}_{X_l}$. Therefore $\widetilde{\mathcal{F}}^1$ is dual to $R^1 f_* \mathcal{O}_{X_l}$. The duality is induced by a relatively ample line bundle on $f_{X_l} : X_l \longrightarrow l$. So

$$\widetilde{\mathcal{F}}^1 \simeq \mathcal{T}_{\mathbb{P}^2|_l}.$$

From the normal bundle sequence

$$0 \longrightarrow \mathcal{T}_{\mathbb{P}^1} \longrightarrow \mathcal{T}_{\mathbb{P}^2|_l} \longrightarrow \mathcal{N}_{l/\mathbb{P}^2} \longrightarrow 0$$

follows that

$$c_1(\mathcal{T}_{\mathbb{P}^2|_l}) = c_1(\mathcal{T}_{\mathbb{P}^1}) + c_1(\mathcal{N}_{l/\mathbb{P}^2}) = 3.$$

Now theorem 3.7 implies the claim.

□

Theorem 3.15. *Let $f : X \longrightarrow \mathbb{P}^2$ be a projective Lagrangian fibration with principally polarised fibres and unipotent monodromy of rank one. Let $l \subset \mathbb{P}^2$ be a general line. Then*

$$\deg(\Delta) = 30 + 2\gamma^*(\mu_l),$$

where μ_l denotes a monodromy factorisation of f_{X_l}.

Proof: The algebraic degree of Δ is $\deg(\Delta) = l \cdot \Delta = \deg(\mu_l)$. Corollary 3.12 and Theorem 3.14 imply the claim.

□

3.3. **The geometrical interpretation of the generator γ^*.** We now give a geometrical interpretation of the class $\gamma^* \in H^1(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$. In order to do that we have to describe a compactification of the moduli space of principally polarised abelian surfaces.

The moduli space of principally polarised abelian surfaces $\mathcal{A}^2$ is the quotient

$$\mathcal{A}^2 = \mathfrak{h}_2/\operatorname{Sp}(4,\mathbb{Z})$$

of the Siegel upper half-space by the action of the integral symplectic group. The moduli space $\mathcal{A}_2$ is closely related to the moduli space of smooth genus two curves $\mathcal{M}_2$. This space is the quotient

$$\mathcal{M}_2 = \mathcal{T}_2/\operatorname{Map}_2$$

of the Teichmüller space $\mathcal{T}_2$ by the mapping class group. The Deligne-Mumford compactification $\overline{\mathcal{M}}_2$ is the moduli space of stable genus two curves, [20]. The boundary $\overline{\mathcal{M}}_2 \setminus \mathcal{M}_2$ consists of two divisors D_0, D_1.

Definition 3.16.

i) *Let $D_0 \subset \overline{\mathcal{M}}_2$ be the divisor parametrising curves with (at least) one non-separating node.*
ii) *Let $D_1 \subset \overline{\mathcal{M}}_2$ be the divisor parametrising curves with one separating node.*

A general point in D_0 is represented by an irreducible genus two curve with a single non-separating node. A point in $D_1 \setminus D_0$ on the other hand is represented by a genus two curve with a single separating node. Such a curve consists in two elliptic curves E_1 and E_2 that intersect in one point.

Associated to a smooth curve C is its Jacobian $\mathrm{Jac}(C)$. This is naturally a principally polarised abelian variety. A principal polarisation ω of an abelian variety A defines (up to translation) a divisor on A, the so called theta divisor Θ. Fixing a point c_0 on the curve C leads to an embedding $\phi : C \longrightarrow \mathrm{Jac}(C)$ and more generally to maps $\phi^{(n)} : C^{(n)} \longrightarrow \mathrm{Jac}(C)$ and to subvarieties $W_n := \operatorname{im}\phi^{(n)}$. W_{g-1} is then the theta divisor. In case $g = 2$, C is thus isomorphic to the theta divisor. The map

$$\begin{aligned} \mathrm{Jac} : \mathcal{M}_2 &\longrightarrow \mathcal{A}_2 \\ C &\mapsto (\mathrm{Jac}(C), [W_1]) \end{aligned}$$

is holomorphic and injective by the classical Torelli theorem. Moreover with respect to the orbifold structures

$$\mathcal{M}_2 = \mathcal{T}_2/\operatorname{Map}_2\ ; \quad \mathcal{A}^2 = \mathfrak{h}_2/\operatorname{Sp}(4,\mathbb{Z})$$

this map is a morphism of orbifolds, i.e. it lifts locally to a map from $\mathcal{T}_2$ to $\mathfrak{h}_2$ that is equivariant with respect to the homomorphism $\xi : \operatorname{Map}_2 \longrightarrow \operatorname{Sp}(4,\mathbb{Z})$, see [56]. As we said, a point in $D_1 \setminus D_0$ corresponds to a curve $E_1 \cup E_2$ with E_1, E_2 two elliptic curves and $E_1 \cap E_2$ a point. Therefore

$$D_1 \setminus D_0 = \mathcal{M}_1^{(2)}.$$

Associating to such a curve the Jacobian of its normalisation, i.e. $\operatorname{Jac}(E_1)\times\operatorname{Jac}(E_2)$ together with the natural polarisation extends Jac to a holomorphic map

$$\operatorname{Jac} : \overline{\mathcal{M}}_2 \setminus D_0 \longrightarrow \mathcal{A}_2.$$

This is an isomorphism of orbifolds, as can be seen from the following decomposition, see [56]

$$\overline{\operatorname{Jac}(\mathcal{M}_2)} = \left\{ \begin{array}{ll} (A,\Theta)| & A = \operatorname{Jac}(C) \text{ with } C \text{ smooth of genus } 2 \\ & \text{and } \Theta \text{ the corresponding theta divisor or} \\ & A = \operatorname{Jac}(E_1)\times\operatorname{Jac}(E_2) \text{ with } E_i \text{ smooth of genus } 1 \\ & \text{and } \Theta = \Theta_1 \times \operatorname{Jac}(E_2) \cup \operatorname{Jac}(E_1)\times\Theta_2 \end{array} \right\}.$$

As $\mathcal{M}_2$ and $\mathcal{A}_2$ have the same dimension and $\mathcal{A}_2$ is irreducible, $\overline{\operatorname{Jac}(\mathcal{M}_2)} = \mathcal{A}_2$. Thus $\operatorname{Jac}(\mathcal{M}_2)$ consists of those abelian varieties that have irreducible theta divisor. The Siegel upper half-space $\mathfrak{h}_2$ contains the space

$$\mathfrak{h}_1 \times \mathfrak{h}_1 = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \mid \operatorname{Im} z_i > 0,\ i = 1,2 \right\}$$

of diagonal matrices. The element $(a_1 a_2)^3$ of Map_2 maps under $\xi : \operatorname{Map}_2 \longrightarrow \operatorname{Sp}(4,\mathbb{Z})$ to the following matrix in $\operatorname{Sp}(4,\mathbb{Z})$,

$$\sqrt{\gamma} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The fix point set of $\sqrt{\gamma}$ is exactly $\mathfrak{h}_1 \times \mathfrak{h}_1$. The stabiliser of $\mathfrak{h}_1 \times \mathfrak{h}_1$ is given by the normaliser $N(\pm\sqrt{\gamma})$, see [14]. Denote by U the orbit $\operatorname{Sp}(4,\mathbb{Z}) \cdot \mathfrak{h}_1 \times \mathfrak{h}_1$ and by V

the complement of U in $\mathfrak{h}_2$, i.e.

$$U = \bigcup_{g \in Sp(4,\mathbb{Z})} g(\mathfrak{h}_1 \times \mathfrak{h}_1), \qquad V = \mathfrak{h}_2 \setminus \bigcup_{g \in Sp(4,\mathbb{Z})} g(\mathfrak{h}_1 \times \mathfrak{h}_1).$$

This is an invariant decomposition and so it induces a decomposition of $\mathcal{A}_2$. From the above

$$\mathrm{Jac}(D_1 \setminus D_0) = U/\operatorname{Sp}(4,\mathbb{Z}) \ ; \quad \mathrm{Jac}(\mathcal{M}_2) = V/\operatorname{Sp}(4,\mathbb{Z}).$$

The subgroup $N(\pm\sqrt{\gamma})$ is isomorphic to $(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})) \ltimes \mathbb{Z}_2$ and therefore

$$\begin{aligned} U/\operatorname{Sp}(4,\mathbb{Z}) &= (\mathfrak{h}_1 \times \mathfrak{h}_1)/N(\pm\sqrt{\gamma}) \\ &= \mathcal{A}_1^{(2)}. \end{aligned}$$

This implies that $\mathrm{Jac}_{|D_1 \setminus D_0} : D_1 \setminus D_0 \longrightarrow U/\operatorname{Sp}(4,\mathbb{Z})$ is an orbifold isomorphism and thereby that $\mathrm{Jac} : \overline{\mathcal{M}}_2 \setminus D_0 \longrightarrow \mathcal{A}_2$ is an orbifold isomorphism. Furthermore $\mathrm{Jac} : \overline{\mathcal{M}}_2 \setminus D_0 \longrightarrow \mathcal{A}_2$ extends to an isomorphism

$$\overline{\mathrm{Jac}} : \overline{\mathcal{M}}_2 \longrightarrow \overline{\mathcal{A}}_2,$$

where $\overline{\mathcal{A}}_2$ is the Igusa compactification of $\mathcal{A}_2$, see [34] and [59]. We will use the letters D_0, D_1 also to denote the corresponding divisors in $\overline{\mathcal{A}}_2$.

Definition 3.17. *A principally polarised abelian surface is reducible as a p.p.a.s. if it is a product of two elliptic curves with the natural polarisation, i.e. if it represents a point in D_1.*

The boundary $\overline{\mathcal{A}}_2 \setminus \mathcal{A}_2$ has one component, namely D_0. The points of $\overline{\mathcal{A}}_2 \setminus \mathcal{A}_2$ correspond to compactified generalised Jacobians of stable curves. We briefly describe the compactified generalised Jacobian of a general member of D_0, see [62], p. 83. The Jacobian of a smooth genus two curve C can be defined as $\mathrm{Pic}^0(C)$. The canonical class K_C induces a natural isomorphism $\mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^2(C)$. Consider the natural map

$$C^{(2)} \longrightarrow \mathrm{Pic}^2(C)$$

that sends each degree two cycle to the corresponding line bundle. Let $\phi_{K_C} : C \longrightarrow \mathbb{P}^1$ be the natural degree two map. The pairs of the form $\phi_{K_C}^{-1}(x)$ for $x \in \mathbb{P}^1$ are then all linearly equivalent. Conversely $C^{(2)}$ is the blow up of the point K_C in $\mathrm{Pic}^2(C)$. Consider now C a general element of D_0. The normalisation of C

$$\mathrm{norm} : \mathrm{E} \longrightarrow \mathrm{C}$$

is then an elliptic curve E. Let $p_1, p_2 \in E$ be the two points lying over the node q of C. The generalised Jacobian of C is a $\mathbb{C}^*$-bundle

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(E) \longrightarrow 0$$

over $\mathrm{Jac}(E)$. The reason for this is that a degree zero line bundle on the nodal curve C is given by a degree zero line bundle on E together with an identification of the fibres L_{p_1} and L_{p_2}. The natural compactification of this $\mathbb{C}^*$-bundle is a $\mathbb{P}^1$-bundle over $\mathrm{Jac}(E)$ obtained by adding a zero-section σ_0 and an infinity-section σ_∞. The compactified Jacobian $\overline{\mathrm{Jac}}(C)$ of C is then constructed by identifying these two sections over a translation t in $\mathrm{Jac}(E)$, i.e. $\sigma(x)$ is glued to $\sigma(x+t)$, where t is the line bundle $\mathcal{O}_E(p_1 - p_2)$.

Let now $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface and discriminant locus $\Delta = \{p_1, ..., p_d\}$. What we have in mind is again the restriction of a Lagrangian fibration to a general line in $\mathbb{P}^2$. This defines a moduli map

$$\varphi : \mathbb{P}^1 \setminus \Delta \longrightarrow \mathcal{A}_2. \tag{34}$$

To the family $f_{X_{\mathbb{P}^1\setminus\Delta}} : X_{\mathbb{P}^1\setminus\Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$ of abelian surfaces corresponds a family $\mathfrak{f} : \overline{Y}_{\mathbb{P}^1\setminus\Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$ of stable genus two curves, such that the relative Jacobian of $\mathfrak{f}$ is locally over $\mathbb{P}^1 \setminus \Delta$ isomorphic to $f_{X_{\mathbb{P}^1\setminus\Delta}} : X_{\mathbb{P}^1\setminus\Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$.

Lemma 3.18. *Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface. Then there exists a family of stable genus two curves $\mathfrak{f} : \overline{Y}_{\mathbb{P}^1\setminus\Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$ such that the relative Jacobian of $\mathfrak{f}$ is locally over $\mathbb{P}^1 \setminus \Delta$ isomorphic to $f_{X_{\mathbb{P}^1\setminus\Delta}} : X_{\mathbb{P}^1\setminus\Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$.*

Proof: The fibration $f_{X_{\mathbb{P}^1\setminus\Delta}} : X_{\mathbb{P}^1\setminus\Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$ is smooth family of abelian surfaces with a family of principal polarisations ω. The relative Picard variety $\mathrm{Pic}^\omega(X_{\mathbb{P}^1\setminus\Delta}/\mathbb{P}^1\setminus\Delta)$ is then a smooth family of abelian surfaces with fibre $\mathrm{Pic}^{\omega_b}(X_b)$. $\mathrm{Pic}^{\omega_b}(X_b)$ parametrises line bundles on X_b with first Chern class ω_b. Let e be a point on X_b. Then there exists a universal line bundle $\mathcal{L}$ over $X_b \times \mathrm{Pic}^{\omega_b}(X_b)$ such that $\mathcal{L}|X_b \times \{L\} \simeq L$ for all $L \in \mathrm{Pic}^{\omega_b}(X_b)$ and $\mathcal{L}|\{e\} \times \mathrm{Pic}^{\omega_b}(X_b)$ is trivial, see [24] p. 328. This universal line bundle extends locally away from the singular fibres of f. Let therefore U be an open subset of $\mathbb{P}^1 \setminus \Delta$ such that $g : \mathrm{Pic}^\omega(X_U/U) \longrightarrow U$ and $f : X_U \longrightarrow U$ have sections σ and ϵ respectively and let $\mathcal{L}$ be the universal line bundle on $X \times_U \mathrm{Pic}^\omega(X_U/U)$.

$$\begin{tikzcd} X_U \arrow[d, "f"] & X \times_U \mathrm{Pic}^\omega(X_U/U) \arrow[l] \arrow[d] \\ U & \mathrm{Pic}^\omega(X_U/U) \arrow[l, "g"'] \end{tikzcd}$$

The section σ induces a map $\tau := \mathrm{id}_{X_U} \times (\sigma \circ f) : X_U \longrightarrow X \times_U \mathrm{Pic}^\omega(X/U)$. For each $b \in U$ the restriction of the line bundle $\tau^*\mathcal{L}$ to the fibre X_b has $h^0(X_b, \tau^*\mathcal{L}_{|X_b}) = 1$ as it is a principal polarisation. The direct image $f_*\tau^*\mathcal{L}$ is therefore a line bundle on U by Theorem 8.5 in [3]. Let t be a non-vanishing section in $f_*\tau^*\mathcal{L}$. Then $t \in H^0(X_U, \tau^*\mathcal{L})$ is such that the restrictions $t_{|X_b} \in H^0(X_b, \tau^*\mathcal{L}_{|X_b})$ for $b \in U$ are non-trivial sections. Thus $(t)_0$ defines a family of theta divisors over U. As two theta divisors differ by a unique translation, two such local families are canonically isomorphic on their overlap in $\mathbb{P}^1 \setminus \Delta$. Thus one obtains a family of stable genus two curves over $\mathbb{P}^1 \setminus \Delta$. Choosing local sections of $f_{X_{\mathbb{P}^1\setminus\Delta}} : X_{\mathbb{P}^1\setminus\Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$ yields isomorphisms to the relative Jacobian of $\mathfrak{f}$ locally over the base.

$\square$

Consider again the situation of a principally polarised abelian fibration $f : X \longrightarrow \mathbb{P}^1$ with discriminant locus $\Delta = \{p_1, ..., p_d\}$. Let D be a disc in $\mathbb{P}^1$ centred in p_i. Then there is a multi-valued holomorphic map

$$\phi : D^* \longrightarrow \mathfrak{h}_2$$

such that

$$\phi(\alpha \cdot t) = T_i \phi(t) \tag{35}$$

and $\varphi(t) = \phi(t) \bmod \mathrm{Sp}(4, \mathbb{Z})$, where T_i is the monodromy of f around p_i and $\phi(\alpha \cdot t)$ denotes the analytic continuation of $\phi(t)$ around p_i. Assume that the general fibre of $f : X \longrightarrow \mathbb{P}^1$ is not reducible as a p.p.a.s.. Then as D_1 is closed in $\overline{\mathcal{A}}_2$ we can assume $\varphi(D^*) \subset \overline{\mathcal{A}}_2 \setminus D_1$. The map $\varphi : D^* \longrightarrow \mathcal{A}_2$ can be extended to a holomorphic map $\varphi : D \longrightarrow \overline{\mathcal{A}}_2$, see [61], p. 150. Namikawa and Ueno show, [60] Theorem 6 and 7, that there exists a unique relatively minimal genus two fibration $\mathfrak{f} : Y \longrightarrow D$ such that Y and Y_t for $t \neq 0$ are smooth and such that the moduli map defined by $\mathfrak{f}_{|D^*}$ coincides with $\varphi_{|D^*}$. The results of Namikawa and Ueno imply in particular that for a family of genus two curves $\mathfrak{f} : Y \longrightarrow D$ with smooth fibres over D^* the singular fibre is completely determined by $\varphi : D^* \longrightarrow \mathcal{A}_2$. They classify all singular fibres, see [61]. In our case the monodromy is unipotent, which implies that the singular fibre of $\mathfrak{f}$ is a reduced nodal curve, see also [73] Proposition 3. By contracting the (-2)-curves contained in the singular fibre we obtain a surface $\overline{Y}$ with rational double points and a fibration $\mathfrak{f} : \overline{Y} \longrightarrow D$ in stable genus two curves. Thus the family of stable curves $\mathfrak{f} : \overline{Y}_{\mathbb{P}^1 \setminus \Delta} \longrightarrow \mathbb{P}^1 \setminus \Delta$ extends uniquely to a family of stable curves over $\mathbb{P}^1$. Therefore we obtain from $f : X \longrightarrow \mathbb{P}^1$ a family of stable genus two curves $\mathfrak{f} : \overline{Y} \longrightarrow \mathbb{P}^1$. We call $\mathfrak{f}$ the *family of stable genus two curves corresponding to* f and denote the minimal resolution by $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$.

We denote the extention of the map (34) to $\mathbb{P}^1$ by

$$\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathcal{A}}_2.$$

We briefly describe the singular fibres of $\mathfrak{f}$. Let $q_i \in \mathbb{P}^1$ be a point such that $\varphi(q_i) \in D_1$ and let h be a minimal defining equation for D_1 locally around $\varphi(q_i)$. As the general fibre of f is not reducible as a p.p.a.s. $\mathrm{ord}_{q_i}(\varphi^* h)$ is finite. Let the monodromy around $b \in \mathbb{P}^1$ be t^k for a simple transvection t and $k \in \mathbb{N}$ and assume $\mathrm{ord}_b(\varphi^* h) = 0$. The fibre $\overline{Y}_b$ has then a single non-separating node p and the fibre Y_b is $\overline{Y}_b$ where p is replaced by a string of $k-1$ (-2)-curves. The monodromy of $\mathfrak{f}$ around b is then τ^k for τ (the class of) a Dehntwist along a non-separating curve. On the other hand let $\mathrm{ord}_b(\varphi^* h) = m \geq 1$ and the monodromy around b be t^k for $k \in \mathbb{N}_0$. The fibre $\overline{Y}_b$ consists then in two elliptic curves intersecting in one point, if $k = 0$ and in an elliptic curve and a rational curve with one node intersecting in one point, if $k \geq 1$. The fibre Y_b has the separating node replaced by a string of $m-1$ (-2)-curves and the non-separating node by a string of $k-1$ (-2)-curves.

Remark 3.19. Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface, discriminant locus Δ and unipotent monodromy of rank one. Then

$$\deg(\Delta) = \deg(\varphi^* D_0).$$

The geometrical interpretation of γ^* is now as follows.

Theorem 3.20. *Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface and unipotent monodromy of rank one. Assume that the general fibre is not reducible as a p.p.a.s.. Then*

$$\gamma^*(\mu) = -\deg(\varphi^* D_1),$$

where μ is a monodromy factorisation of f.

Proof: Let $\mathfrak{f} : \overline{Y} \longrightarrow \mathbb{P}^1$ be the family of stable genus two curves corresponding to $f : X \longrightarrow \mathbb{P}^1$. The moduli maps form the commutative diagram

$$\begin{tikzcd} & \overline{\mathcal{A}}_2 \arrow[d, "\overline{\mathrm{Jac}}^{-1}"] \\ \mathbb{P}^1 \arrow[ur, "\varphi"] \arrow[r, "\varphi_{\mathfrak{f}}"'] & \overline{\mathcal{M}}_2 \end{tikzcd} .$$

The set $\varphi^{-1}(D_1)$ is finite as the general fibre is not reducible as a p.p.a.s.. The discriminant locus of the abelian fibration is

$$\Delta = \varphi^* D_0.$$

Let $\{p_1, ..., p_d\}$ be the underlying set. The discriminant locus of the genus two fibration $\mathfrak{f} : \overline{Y} \longrightarrow \mathbb{P}^1$ on the other hand is

$$\Delta_{\mathfrak{f}} = \varphi_{\mathfrak{f}}^*(D_0 + D_1).$$

Let $\{p_1, ..., p_d, q_1, ..., q_m\}$ be the underlying set. Choose a base point $b_0 \in \mathbb{P}^1 \setminus \Delta_{\mathfrak{f}}$ and loops $\alpha_1, ..., \alpha_d, \beta_1, ..., \beta_m$ such that α_i goes counterclockwise around the p_i and β_j around q_j and such that

$$\pi_1(\mathbb{P}^1 \setminus \Delta_{\mathfrak{f}}) = \langle \alpha_i, \beta_j | \prod_{i=1}^{d} \alpha_i \cdot \prod_{j=1}^{m} \beta_i = 1 \rangle.$$

Then

$$\pi_1(\mathbb{P}^1 \setminus \Delta) = \langle \alpha_i | \prod_{i=1}^{d} \alpha_i = 1 \rangle.$$

Let

$$\mu : \pi_1(\mathbb{P}^1 \setminus \Delta) \longrightarrow \mathrm{Sp}(4, \mathbb{Z})$$

be the monodromy representation of the abelian fibration and

$$\mu_{\mathfrak{f}} : \pi_1(\mathbb{P}^1 \setminus \Delta_{\mathfrak{f}}) \longrightarrow \mathrm{Map}_2$$

the one of the genus two fibration. They form the commutative diagram

$$\begin{tikzcd} \pi_1(\mathbb{P}^1 \setminus \Delta_{\mathfrak{f}}) \arrow[r, "\mu_{\mathfrak{f}}"] \arrow[d] & \mathrm{Map}_2 \arrow[d, "\xi"] \\ \pi_1(\mathbb{P}^1 \setminus \Delta) \arrow[r, "\mu"] & \mathrm{Sp}(4, \mathbb{Z}) \end{tikzcd} .$$

Consider first the case that $\varphi_{\mathfrak{f}}(\mathbb{P}^1) \cap D_0$ and $\varphi_{\mathfrak{f}}(\mathbb{P}^1) \cap D_1$ are disjoint. Let

$$\mu(\alpha_i) = t_i^{k_i}.$$

be the monodromy transformation of f along α_i, where t_i is a simple transvection. The monodromy transformation of $\mathfrak{f}$ along α_i is then

$$\mu_{\mathfrak{f}}(\alpha_i) = \tau_i^{k_i},$$

where τ_i is a Dehn twist along a simple closed, non-separating curve such that $\xi(\tau_i) = t_i$. The monodromy factorisation of $\mathfrak{f}$ is

$$\mu_{\mathfrak{f}} : \left(\prod_{i=1}^{d} \tau_i^{k_i}\right) \cdot \left(\prod_{j=1}^{m} \mu_{\mathfrak{f}}(\beta_j)\right) = 1. \tag{36}$$

The transformations $\mu_{\mathfrak{f}}(\beta_j)$ lie in the Torelli group $\mathcal{J}_2$ as the monodromy of the abelian fibration along β_j is trivial. Therefore both factors on the left hand side of (36) lie in the Torelli group and we can apply the γ-degree from Definition 2.13 to them. Equation (36) then implies

$$\deg_\gamma \left(\prod_{i=1}^{d} \tau_i^{k_i}\right) = -\deg_\gamma \left(\prod_{j=1}^{m} \mu_{\mathfrak{f}}(\beta_j)\right).$$

The fibre of $\mathfrak{f} : \overline{Y} \longrightarrow \mathbb{P}^1$ over the point q_j is a singular curve with non-singular Jacobian. Thus it represents a point in $D_1 \setminus D_0$, i.e. a curve with a single node that is separating. The fibre in the resolution Y has a string of (-2)-curves in place of this node. The γ-degree $\deg_\gamma(\mu_{\mathfrak{f}}(\beta_j))$ is the number of separating vanishing cycles of $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ in q_j. Therefore $\deg_\gamma(\mu_{\mathfrak{f}}(\beta_j)) = \operatorname{ord}_{q_j}(\varphi^* h_j)$, where h_j is a minimal equation for D_1 locally around $\varphi(q_j)$. Thus

$$\begin{aligned} \deg(\varphi^* D_1) &= \sum_{j=1}^{m} \operatorname{ord}_{q_j}(\varphi^* h_j) \\ &= \sum_{j=1}^{m} \deg_\gamma\left(\mu_{\mathfrak{f}}(\beta_j)\right) \\ &= -\deg_\gamma \left(\prod_{i=1}^{d} \tau_i^{k_i}\right). \end{aligned}$$

Recall the diagram (16)

$$\begin{tikzcd} \mathcal{J}_2 \arrow[r] \arrow[d, "\deg_\gamma"'] & \mathrm{Map}_2 \arrow[r, "\xi"] \arrow[d, "\rho"] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[d, equal] \\ \mathbb{Z} \arrow[r] & G_{\gamma^*} \arrow[r, "p"] & \mathrm{Sp}(4,\mathbb{Z}) \end{tikzcd}.$$

Here the bottom sequence is the extension defined by the class $\gamma^* \in H^2(\mathrm{Sp}(4,\mathbb{Z}),\mathbb{Z})$. We claim that the Dehn twist τ_i maps under ρ to the distinguished lift of t_i in G_{γ^*}. This can be seen as follows. As τ_i is a Dehn twist along a non-separating curve, it is conjugate to the Dehn twist $a_2^{\mathrm{Map}_2}$, i.e. $\tau_i = \delta * a_2^{\mathrm{Map}_2}$ for $\delta \in \mathrm{Map}_2$, where $a_2^{\mathrm{Map}_2}$ denotes the image of $a_2 \in \mathrm{Br}_6$ in Map_2. Likewise G_{γ^*} is a quotient of Br_6 and we denote the image of $a_2 \in \mathrm{Br}_6$ in G_{γ^*} by $a_2^{G_{\gamma^*}}$. Then $\rho(\tau_i) = \rho(\delta) * \rho(a_2^{\mathrm{Map}_2}) = \rho(\delta) * a_2^{G_{\gamma^*}}$. On the other hand $t_i = \xi(\delta) * a_2^{\mathrm{Sp}(4,\mathbb{Z})}$. The distinguished lift of t_i in $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$ is $\widehat{t_i} = \beta * a_2^{\widehat{\mathrm{Sp}}(4,\mathbb{Z})}$ where β is an arbitrary lift of $\xi(\delta)$ in $\widehat{\mathrm{Sp}}(4,\mathbb{Z})$. As there is a commutative diagram

$$\begin{tikzcd} & \widehat{\mathrm{Sp}}(4,\mathbb{Z}) \arrow[dl] \arrow[d] \\ G_{\gamma^*} \arrow[r, "p"] & \mathrm{Sp}(4,\mathbb{Z}) \end{tikzcd},$$

we can choose β to be a lift of $\rho(\delta)$. This implies that the distinguished lift of t_i to G_{γ^*} is given by $\rho(\delta) * a_2^{G_{\gamma^*}}$.

Denote the distinguished lift of t_i to G_{γ^*} also by $\widehat{t_i}$. Then

$$\deg_\gamma\left(\prod_{i=1}^{d} \tau_i^{k_i}\right) = \gamma^*(\prod_{i=1}^{d} \hat{t}_i^{k_i})$$

and therefore

$$\gamma^*(\mu) = -\deg(\varphi^* D_1).$$

Consider the case that there exists p_i such that $\varphi_{\mathfrak{f}}(p_i)$ lies in $D_0 \cap D_1$ and denote the set of these indices by I. In this case the fibre of $\mathfrak{f} : \overline{Y} \longrightarrow \mathbb{P}^1$ over p_i is a curve $E_1 \cup E_2$ with $E_1 \cap E_2$ a point. As the monodromy is unipotent of rank one, E_1 is a rational curve with one node and E_2 an elliptic curve. The fibre in Y has the two nodes replaced by strings of (-2)-curves. As the vanishing cycles to different nodes of this fibre are disjoint, the Dehn twists along them commute. The monodromy transformation along α_i is therefore

$$\mu_{\mathfrak{f}}(\alpha_i) = \tau_i^{k_i} \cdot \sigma_i,$$

where σ_i is an element of the Torelli group and τ_i a Dehn twist along a simple closed, non-separating curve such that $\xi(\tau_i) = t_i$. For $i \notin I$, $\sigma_i = 1$. The monodromy factorisation of $\mathfrak{f}$ is thus

$$\mu_{\mathfrak{f}} : \left(\prod_{i=1}^{d} \tau_i^{k_i}\sigma_i\right) \cdot \left(\prod_{j=1}^{m} \mu_{\mathfrak{f}}(\beta_j)\right) = 1.$$

The product $\prod_{i=1}^{d} \tau_i^{k_i}\sigma_i$ lies in the Torelli group. Therefore

$$\begin{aligned} 0 &= \deg_\gamma(\left(\prod_{i=1}^{d} \tau_i^{k_i}\sigma_i\right) \cdot \left(\prod_{j=1}^{m} \mu_{\mathfrak{f}}(\beta_j)\right)) \\ &= \deg_\gamma(\left(\prod_{i=1}^{d} \tau_i^{k_i}\sigma_i\right)) + \sum_{j=1}^{m} \deg_\gamma(\mu_{\mathfrak{f}}(\beta_j)). \end{aligned}$$

For $\sigma \in \mathcal{J}_2$

$$\tau_i^{k_i}\sigma\tau_{i+1}^{k_{i+1}} = \tau_i^{k_i}\tau_{i+1}^{k_{i+1}}\sigma'$$

where $\sigma' = \tau_{i+1}^{-k_{i+1}}\sigma\tau_{i+1}^{k_{i+1}}$. And $\deg_\gamma(\sigma') = \deg_\gamma(\sigma)$. This implies

$$0 = \deg_\gamma(\left(\prod_{i=1}^{d} \tau_i^{k_i}\right)) + \sum_{i=1}^{d} \deg_\gamma(\sigma_i) + \sum_{j=1}^{m} \deg_\gamma(\mu_{\mathfrak{f}}(\beta_j)) \tag{37}$$

Now $\mathrm{ord}_{p_i}(\varphi^* h_i) = \deg_\gamma(\sigma_i)$ and thus the equality

$$\deg(\varphi^* D_1) = -\deg_\gamma\left(\prod_{i=1}^{d} \tau_i^{k_i}\right)$$

is valid in this case too.

□

From Corollary 3.12 we get.

Corollary 3.21. *Let $f : X \longrightarrow \mathbb{P}^1$ be a flat morphism with general fibre a principally polarised abelian surface, discriminant locus Δ and unipotent monodromy of rank one. Assume that the general fibre is not reducible as a p.p.a.s.. Then*

$$\deg(\Delta) = 10c_1(\widetilde{\mathcal{F}}^1) - 2\deg(\varphi^* D_1).$$

□

Remark 3.22. Theorem 3.20 and Corollary 3.21 remain valid in case the monodromy around each point p_i is a product of two commuting transvections. The proofs are strictly analogue.

3.4. **A formula for** $\deg(\Delta)$ **and an upper bound for** $\deg(\Delta)$**.** The geometrical interpretations of $\kappa^* + \gamma^*$ and γ^* together with Theorem 3.15 give us the following formula. For $f : X \longrightarrow \mathbb{P}^2$ a Lagrangian fibration with principally polarised fibres and l a general line in $\mathbb{P}^2$ we denote the extension of moduli map of $f_{X_{l\setminus\Delta}} : X_{l\setminus\Delta} \longrightarrow l \setminus \Delta$ to l by $\varphi : l \longrightarrow \overline{\mathcal{A}}_2$.

Theorem 3.23. *Let $f : X \longrightarrow \mathbb{P}^2$ be a projective Lagrangian fibration with principally polarised fibres, discriminant locus Δ and unipotent monodromy of rank one. Assume that the general fibre is not reducible as a p.p.a.s.. Then*

$$\deg(\Delta) = 30 - 2\,\deg(\varphi^* D_1).$$

Proof: The claim follows from Theorem 3.20 and Theorem 3.15.

□

Example 3.24. (Compactified Jacobian of a linear system on a $K3$-surface) Let R be a general sextic in $\mathbb{P}^2$. The double cover of $\mathbb{P}^2$ branched along R is then a $K3$-surface S,

$$\pi : S \xrightarrow{2:1} \mathbb{P}^2.$$

The inverse image of a general line $L \subset \mathbb{P}^2$ is a double cover of $\mathbb{P}^1$ branched in 6 points and consequently a smooth genus two curve. Lines in $\mathbb{P}^2$ are parametrised by the dual $\mathbb{P}^{2*}$. If L^* is the point in $\mathbb{P}^{2*}$ that corresponds to the line L in $\mathbb{P}^2$, we say that L^* is *dual* to L and vice-versa. The correspondence between $\mathbb{P}^2$ and $\mathbb{P}^{2*}$ is such that the line x^* in $\mathbb{P}^{2*}$ dual to $x \in \mathbb{P}^2$ parametrises the pencil of lines through x. The curves $\pi^{-1}(L)$ form a family of genus two curves over the dual $\mathbb{P}^{2*}$,

$$\mathfrak{f} : \mathcal{C} \longrightarrow \mathbb{P}^{2*}.$$

The singular curves in this family are parametrised by the plane curve R^* dual to R. Under the duality of plane curves simple double tangents of R correspond to nodes of R^* and simple inflection tangents of R to cusps of R^*. The fact that R is a general sextic implies that it is a Plücker curve, i.e. it has only simple double tangents and simple inflection tangents. Equivalently R^* has only nodes and cusps as singularities. According to Plücker's formulas, the dual curve R^* is an irreducible curve of degree 30. Let l be a general line in $\mathbb{P}^{2*}$. Then l parametrises the pencil of lines in $\mathbb{P}^2$ through a point x and it intersects R^* transversally in 30 distinct points. Let $y \in l \cap R^*$ be one of these points. The monodromy of the genus two fibration $\mathfrak{f}_{|\mathcal{C}_l} : \mathcal{C}_l \longrightarrow l$ around y is easily calculated. Consider a small circle c around y in l. The point y corresponds to a simple tangent y^* to R and the lines parametrised by c are not tangent to R. Choose a line L' in $\mathbb{P}^2$ not containing x that intersects R in six distinct points. Then consider the projection out of x onto L' $\mathrm{pr} : \mathbb{P}^2 \setminus \{x\} \longrightarrow L'$. We can identify c with a small circle around $o := \mathrm{pr}(y^* \setminus \{x\})$ in L'. The vertical lines $\mathrm{pr}^{-1}(t)$ for $t \in c$ intersect R in six distinct points so that moving once around o we get a braid. The relevant braid monodromy is explained for example in [71], section VI. The projection $\mathrm{pr}_{|R} : R \longrightarrow L'$ has a simple smooth critical point over o, therefore the braid along c is a half twist. Consequently the monodromy of $\mathfrak{f}_{|\mathcal{C}_l} : \mathcal{C}_l \longrightarrow l$ around y is a Dehn twist along a non-separating curve and the singular fibre an irreducible curve with a single node. Thus the monodromy over a general line factorises in 30 non-separating Dehn twists.

The singular fibre over a node y of R^* is the double cover of a simple double tangent y^* to R. For a general line l through y we do the same braid monodromy calculation as above. In this case the projection $\mathrm{pr}_{|R} : R \longrightarrow L'$ has two simple smooth critical points over o. Consequently the singular fibre is irreducible with two nodes.

Finally for a cusp y in R^* the line y^* is a simple inflection tangent to R that intersects R in three more points. Number the six points in $\mathrm{pr}^{-1}(t) \cap R$ such that the first three points come together in the inflection point. Then the braid is the product of the two half twists a_1 and a_2. This implies that the fibre over y is irreducible with one cusp.

As all fibres of $\mathfrak{f} : \mathcal{C} \longrightarrow \mathbb{P}^{2*}$ are reduced and irreducible one can construct the relative compactified Jacobian $X := \overline{\mathrm{Jac}}(\mathcal{C})$. This space is smooth and admits a holomorphic-symplectic structure, see [6]. Moreover X is birational to $S^{[2]}$. By a result of Huybrechts [32] it is then deformation equivalent to $S^{[2]}$. The natural map $f : \overline{\mathrm{Jac}}(\mathcal{C}) :\longrightarrow \mathbb{P}^{2*}$ is a Lagrangian fibration.
Let l be a general line in $\mathbb{P}^{2*}$. As the monodromy of $\mathfrak{f}_{|\mathcal{C}_l} : \mathcal{C}_l \longrightarrow l$ factorises in 30 non-separating Dehn twists, the monodromy of $f_{|\overline{\mathrm{Jac}}(\mathcal{C})_l} : \overline{\mathrm{Jac}}(\mathcal{C})_l \longrightarrow l$ factorises in 30 simple transvections. The discriminant locus Δ of f is therefore R^* and has

$$\deg(\Delta) = 30.$$

We saw that $\mathcal{C}_l$ contains only irreducible curves. From this we deduce

$$\deg(\varphi^* D_1) = 0.$$

So our formula is satisfied in this example.

Theorem 3.23 gives an upper bound on the degree of the discriminant locus, except for a very special case of Lagrangian fibration.

Corollary 3.25. *Let $f : X \longrightarrow \mathbb{P}^2$ be a projective Lagrangian fibration with principally polarised fibres, discriminant locus Δ and unipotent monodromy of rank one. Assume that the general fibre is not reducible as a p.p.a.s.. Then*

$$\deg(\Delta) \leq 30$$

Proof: If the general fibre is not a product, then the image of $\mathbb{P}^2 \setminus \Delta$ under the moduli map to $\mathcal{A}_2$ is not contained in D_1. So for l a general line in $\mathbb{P}^2$, $\deg(\varphi^* D_1)$ is non-negative and the claim follows from Theorem 3.23.

□

Remark 3.26. Let $f : X \longrightarrow \mathbb{P}^2$ be a Lagrangian fibration such that the general fibre is a product of two elliptic curves and the general singular fibre is semi-stable and assume that f admits a section. For such a fibration Kamenova proved in [36], Theorem 5.2, that X is deformation equivalent to the Hilbert square $S^{[2]}$ of a $K3$-surface. For such a fibration the section induces a fibrewise principal polarisation such that the general fibre is reducible as a p.p.a.s..

3.5. **Further restrictions on the values of** $\deg(\Delta)$ **and** γ^***.** In this section we prove further restrictions on the values of deg and γ^*.

Let $f : X \longrightarrow \mathbb{P}^2$ be a projective Lagrangian fibration with principally polarised fibres, discriminant locus Δ, and unipotent monodromy of rank one. Assume that the general fibre is not reducible as a principally polarised abelian variety. For $l \subset \mathbb{P}^2$ a general line, we set

$$n := \deg(\Delta) = \deg(\varphi^* D_0); \quad s := -\gamma^*(\mu_l) = \deg(\varphi^* D_1),$$

where φ denotes the moduli map $\varphi : l \longrightarrow \overline{\mathcal{A}}_2$. By Theorem 3.23

$$n + 2s = 30.$$

We claim

Theorem 3.27. *Let $f : X \longrightarrow \mathbb{P}^2$ be a projective Lagrangian fibration with principally polarised fibres and unipotent monodromy of rank one. Assume that the* *general fibre is not reducible as a p.p.a.s.. Then*

$$n \geq 10 \quad (s \leq 10)$$

and s is even, i.e. the possibilities for (n, s) are $(30, 0), (26, 2), (22, 4), (18, 6), (14, 8)$ *and* $(10, 10)$.

In order to prove the theorem we need a couple of lemmata. Let $f : X_l \longrightarrow \mathbb{P}^1$ be the restriction of f to l. According to Lemma 3.18 and the discussion thereafter there exists a family of stable genus two curves $\mathfrak{f} : \overline{Y} \longrightarrow \mathbb{P}^1$ corresponding to $f : X_l \longrightarrow \mathbb{P}^1$. The fibration $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ on the minimal resolution of $\overline{Y}$ then contains s separating and n non-separating nodes.

Definition 3.28. *If a genus two fibration $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ arises in this way, we say that it comes from a Lagrangian fibration.*

In the sequel we are going to study these surfaces. The first thing we note is that Y is projective, as it carries a genus two fibration. Let F be a fibre. Then the adjunction formula gives

$$2 = F(F + K_Y) = F \cdot K_Y.$$

Thus $dF + K$ will have positive selfintersection for d sufficiently large. By Theorem 6.2 in [3] Y is then projective. For the background on genus two fibrations we refer to [29] and [78]. For a surface Y one has the Noether formula:

$$12\chi(\mathcal{O}_Y) = K_Y^2 + \chi_{\text{top}}(Y)$$

where $\chi_{\text{top}}(Y)$ is the topological Euler characteristic. Let $\mathfrak{f} : Y \longrightarrow C$ be a fibration with general fibre a smooth genus g curve over a base C of genus b. Then the following formula holds, see [78], p.6

$$\chi(\mathcal{O}_Y) = \deg(\mathfrak{f}_* K_{Y/C}) + (b-1)(g-1), \tag{38}$$

where $K_{Y/C} = K_Y \otimes \mathfrak{f}^* K_C^{-1}$ the dualising sheaf of $\mathfrak{f}$. The relative dualising sheaf $\mathfrak{f}_* K_{Y/C}$ is then a vector bundle of rank g. Further the topological index theorem states

$$\tau(Y) = \frac{1}{3}(K_Y^2 - 2\chi_{\text{top}}(S)),$$

where $\tau(Y) = b_2^+ - b_2^-$ is the signature of Y, see [3].

In case of a genus two fibration $\mathfrak{f} : Y \longrightarrow C$, the sheaf $R^1\mathfrak{f}_*\mathcal{O}_Y$ is locally free of rank two. There is a rational map onto the projectification $\mathbb{P}(R^1\mathfrak{f}_*\mathcal{O}_Y)$ of the bundle $R^1\mathfrak{f}_*\mathcal{O}_Y$, which is the dual of $\mathfrak{f}_*K_{Y/C}$,

$$\begin{tikzcd} Y \arrow[rr, dashed, "\phi"] \arrow[dr] & & \mathbb{P}(R^1\mathfrak{f}_*\mathcal{O}_Y) \arrow[dl] \\ & C & \end{tikzcd}$$

see [78], p.7. The bundle $\mathbb{P}(R^1\mathfrak{f}_*\mathcal{O}_Y)$ has a section Σ such that the self intersection Σ^2 is minimal. Then $e := -\Sigma^2$ defines an invariant of the fibration.

Let now $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ be a genus two fibration that comes from a Lagrangian fibration $f : X \longrightarrow \mathbb{P}^2$. The smooth part $\mathfrak{f}_{\mathbb{P}^1\setminus\Delta_\mathfrak{f}} : Y\setminus\Delta_\mathfrak{f} \longrightarrow \mathbb{P}^1\setminus\Delta_\mathfrak{f}$ of $\mathfrak{f}$ gives rise to a variation of Hodge structure of weight one. This variation of Hodge structure is polarised by the cup product pairing. This polarised variation of Hodge structure is isomorphic to the polarised variation of Hodge structure that comes from the restriction $f_{l\setminus\Delta_\mathfrak{f}} : X_{l\setminus\Delta_\mathfrak{f}} \longrightarrow \mathbb{P}^1 \setminus \Delta_\mathfrak{f}$ of f to $l \setminus \Delta_\mathfrak{f}$, as the two fibrations have the same period map. Denote the canonical extensions of the Hodge bundles of $\mathfrak{f}_{\mathbb{P}^1\setminus\Delta_\mathfrak{f}}$ and $f_{l\setminus\Delta_\mathfrak{f}}$ by $\widetilde{\mathcal{F}}^1_{Y_{\mathbb{P}^1}}$ and $\widetilde{\mathcal{F}}^1_{X_l}$ respectively. Then $\widetilde{\mathcal{F}}^1_{Y_{\mathbb{P}^1}}$ and $\widetilde{\mathcal{F}}^1_{X_l}$ are isomorphic. By a result of Kawamata, Lemma 1 in [38] the relative dualizing sheaf $\mathfrak{f}_*K_{Y/\mathbb{P}^1}$ is the canonical extension $\widetilde{\mathcal{F}}^1_{Y_{\mathbb{P}^1}}$ of the Hodge bundle of $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$. Thus

$$\begin{aligned} \mathfrak{f}_*K_{Y/\mathbb{P}^1} &= \widetilde{\mathcal{F}}^1_{Y_{\mathbb{P}^1}} \\ &\simeq \widetilde{\mathcal{F}}^1_{X_l}. \end{aligned}$$

It follows from the proof of Theorem 3.14 that

$$\widetilde{\mathcal{F}}^1_{X_l} \simeq \mathcal{T}_{\mathbb{P}^2|_l}$$

This implies

$$\mathfrak{f}_*K_{Y/\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

In particular

$$\deg(\mathfrak{f}_*K_{Y/\mathbb{P}^1}) = 3$$

and Formula (38) gives

$$\chi(\mathcal{O}_Y) = 2.$$

We borrow the following Lemma from [10].

Lemma 3.29. *Let $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ be a genus two fibration with s separating nodes. Then*

$$b_2^- \geq 1 + s.$$

Proof: Let $F_j = \sum_{i=1}^{l} C_i$ be a singular fibre of $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ and $D = \sum_i m_i C_i$ for $m_i \in \mathbb{Z}$. By Zariskis Lemma, [3] p. 111. $C_i F_j = 0$ and $D^2 \leq 0$ with $D^2 = 0$ if and only if $D = rF_j$ for $r \in \mathbb{Q}$. I.e. all components but one of F_j generate a $(l-1)$-dimensional negative definite subspace in $H_2(X, \mathbb{Z})$.

The above subspaces for different singular fibres are pairwise orthogonal. Therefore the direct sum of these over all singular fibres is a negative definite subspace in $H_2(X, \mathbb{Z})$ of dimension at least s. This space is still orthogonal to the class of a general fibre F, which has $F^2 = 0$.

□

Lemma 3.30. *Let $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ a genus two fibration that comes from a Lagrangian fibration. Then*

$$K_Y^2 = s - 2.$$

Proof: The Euler characteristic of Y is

$$\chi_{\text{top}}(Y) = \chi_{\text{top}}(\mathbb{P}^1)\chi_{\text{top}}(F_{\text{gen}}) + \sum_j (\chi_{\text{top}}(F_j) - \chi_{\text{top}}(F_{\text{gen}})),$$

where F_{gen} denotes a general fibre and F_j are the singular fibres. Each node augments the Euler characteristic by 1. Therefore

$$\chi_{\text{top}}(Y) = -4 + n + s.$$

As $\chi(\mathcal{O}_Y) = 2$, Noethers formula implies

$$\begin{aligned} K_Y^2 &= 28 - n - s \\ &= s - 2\,, \end{aligned}$$

where in the last line we used $n + 2s = 30$.

□

Next we show that s must be even. For a genus two fibration $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ the rational map $\phi : Y - - > \mathbb{P}(R^1\mathfrak{f}_*\mathcal{O}_Y)$ is generically of degree two and the indeterminacy locus of ϕ is contained in fibres with separating nodes, see [15] p. 12. Let $\widetilde{Y}$ be the blow up of the indeterminacy locus of ϕ and let B be the branch locus of the induced morphism $\widetilde{\phi} : \widetilde{Y} \longrightarrow \mathbb{P}(R^1\mathfrak{f}_*\mathcal{O}_Y)$.

The bundle $\mathbb{P}(R^1\mathfrak{f}_*\mathcal{O}_Y)$ is isomorphic to the Hirzebruch surface $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$. As the fibres of $\mathfrak{f}$ have genus two, the branch locus B is a divisor on $\mathbb{F}_e$ that is linearly equivalent to $6H + mF$, where H is the class of a section such that $H^2 = e$ and F is the class of a fibre of $\pi : \mathbb{F}_e \longrightarrow \mathbb{P}^1$. We determine m.

Siebert and Tian study double covers of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ in [69]. For branch loci B they define a virtual number of critical values μ_{virt} of the projection $\pi_{|B} : B \longrightarrow \mathbb{P}^1$ and relate this to the homology class of B, see [69] p. 262. Different critical points of the projection $\pi_{|B} : B \longrightarrow \mathbb{P}^1$ contribute differently to μ_{virt}. The contribution of a simple smooth critical point of $\pi_{|B} : B \longrightarrow \mathbb{P}^1$ to μ_{virt} is $+1$ and that of a double point of B of type A_k is $k+1$. A simple infinitely close triple point is a triple point with three pairwise tangent branches that form an ordinary triple point after one blow up. A simple infinitely close triple point of B contributes $+12$ to μ_{virt}, as can be seen by deforming it to twelve smooth critical points. The simplest singular fibre of $\mathfrak{f}$ has a single non-separating node. As the total space Y is smooth, the monodromy around such a fibre is a non-separating Dehn twist. A fibre with a single non-separating node arises from a fibre of $\pi : \mathbb{F}_e \longrightarrow \mathbb{P}^1$ where B has a smooth critical point and four points in which B intersects the fibre transversally. Thus the contribution of such a fibre to μ_{virt} is $+1$. This generalises in the following way. Each non-separating node contributes $+1$ to μ_{virt}. A separating node on the other hand corresponds to a simple infinitely close triple point of B, see [78] p. 9 or [69] p. 261. Thus the contribution of a separating node to μ_{virt} is $+12$.

For $B \sim dH + mF$, where H is the homology class of a section such that $H^2 = e$ and F the class of a fibre, Siebert and Tian show (Proposition 1.1 in [69]) that

$$\mu_{\text{virt}} = 2(d-1)m + ed(d-1).$$

Lemma 3.31. *Let $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ a genus two fibration that comes from a Lagrangian fibration. Then $q(Y) = 0$, $p_g(Y) = 1$ and*

$$m = s.$$

In particular s is even.

Proof: $\mathbb{P}(R^1\mathfrak{f}_*\mathcal{O}_Y) \simeq \mathbb{F}_1$, so $e = 1$. As $d = 6$, $\mu_{\text{virt}} = 10m + 30$. On the other hand $\mu_{\text{virt}} = n + 12s$. Now $n + 2s = 30$ implies

$$m = s.$$

In particular s has to be even, as B is the branch divisor of a double covering. Furthermore by Theorem 2.2 in [78] $e = 1$ implies that $q(Y) = 0$ and consequently $p_g(Y) = 1$.

□

Lemma 3.32. *Let $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ a genus two fibration that comes from a Lagrangian fibration. Then $n \geq 10$ (and $s \leq 10$).*

Proof: $K_Y^2 = s - 2$. On the other hand we have by the topological index theorem

$$\begin{aligned} K_Y^2 &= 3\tau(Y) + 2\chi_{\text{top}}(Y) \\ &= 5\chi_{\text{top}}(Y) - 6 + 6b_1 - 6b_2^- \\ &= 5n + 5s - 26 - 6b_2^- \\ &\leq 5n - s - 32, \end{aligned}$$

where we used that $b_1 = 0$, as $q(Y) = 0$, and in the last line Lemma 3.29. Thus $2s \leq 5n - 30$ and together with $30 = n + 2s$ this implies $s \leq n$. Therefore $n \geq 10$ and $s \leq 10$.

□

Proof of the Theorem: Theorem 3.27 now follows from Lemma 3.32 and Lemma 3.31.

□

3.6. **Classification of surfaces, fibred by genus two curves.** In this section we determine the place of the genus two fibred surface Y in the Enriques-Kodaira classification. Let n and s be the numbers defined in the previous section.

Theorem 3.33. *Let $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ be a genus two fibration that comes from a Lagrangian fibration. Then there are the following three possibilities*

(1) *$s = 0$. Then Y is a $K3$-surface S blown up in two points.*
(2) *$s = 2$. Then Y is a minimal surface with Kodaira dimension* $\text{kod}(Y) = 1$ *and $q(Y) = 0$ and $p_g(Y) = 1$.*

(3) $s = 4, 6, 8, 10$. *Then* Y *is a minimal surface of general type with* $q(Y) = 0$ *and* $p_g(Y) = 1$.

Proof: Let S_∞ be the negative section of $\pi : \mathbb{F}_1 \longrightarrow \mathbb{P}^1$. Let H be the class of a section such that $H^2 = 1$ and F be the class of a fibre of π. Then S_∞ is linearly equivalent to $H - F$. The intersection of the branch divisor B with S_∞ is therefore

$$B \cdot S_\infty = (6H + sF) \cdot (H - F) = s. \tag{39}$$

Consider the case $s = 0$. As there are no separating nodes the map ϕ is a morphism $\phi : Y \longrightarrow \mathbb{F}_1$. Over S_∞ it is two to one. The inverse image of S_∞ consists therefore in a pair of disjoint (-1)-curves E_1, E_2. Denote by $\epsilon_1 : Y \longrightarrow T$ be the blow down of E_1 and E_2 and by $p : \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ be the blow down of S_∞ to a point $c \in \mathbb{P}^2$. As B does not intersect S_∞ it projects isomorphically to a curve $\overline{B}$ in $\mathbb{P}^2$ that does not contain the point c. For L a line in $\mathbb{P}^2$

$$\overline{B} \cdot L = B \cdot H = 6.$$

Thus $\overline{B}$ is a plane sextic. The fact that $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ contains only non-separating nodes implies that the only singularities of B are double points. Let $\overline{\phi} : \overline{Y} \longrightarrow \mathbb{F}_1$ be the double cover of $\mathbb{F}_1$ branched along B. Y is then a resolution of $\overline{Y}$. Let $\theta : \overline{S} \longrightarrow \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along $\overline{B}$. Then there is the commutative diagram

$$\begin{tikzcd} \overline{Y} \arrow[r, "\overline{\phi}"] \arrow[d, "\epsilon"] & \mathbb{F}_1 \arrow[d, "p"] \\ \overline{S} \arrow[r, "\theta"] & \mathbb{P}^2 \end{tikzcd}$$

where ϵ is the blow up of the two (smooth) points of $\overline{S}$ lying over c. $\overline{S}$ is a singular $K3$-surface, where by a singular $K3$-surface we mean a surface X with rational singularities, $q(X) = 0$ and $K_X \simeq \mathcal{O}_X$. The minimal resolution of X is then a $K3$-surface. The birational morphism $Y \longrightarrow \overline{S}$ factorises as $Y \xrightarrow{\epsilon_1} T \xrightarrow{\epsilon_2} \overline{S}$. Now $K_Y^2 = -2$ implies $K_T^2 = 0$. Thus T is a $K3$-surface and Y the blow up of a $K3$-surface in two points.

In general we have $\chi(\mathcal{O}_Y) = 2$ and therefore $p_g(Y) \geq 1$. This in turn implies $P_n \geq 1$ for all $n \geq 1$. Therefore $h^0((1-n)K_Y) \leq 1$ for $n > 1$ and Riemann-Roch implies

$$h^0(nK_Y) \geq \chi(\mathcal{O}_Y) - 1 + \left(\frac{n(n-1)}{2}\right) \cdot K_Y^2.$$

In case $s \geq 4$, $K_Y^2 > 0$. This implies $\mathrm{kod}(Y) = 2$. A non-minimal relatively minimal genus two fibration of general type has a unique (-1)-curve and $K_Y^2 = 0$, see [63] p. 20. So in our case Y must be minimal.

Finally in case $s = 2$ we have $K_Y^2 = 0$. As $p_g(Y) = 1$, $\mathrm{kod}(Y) \geq 0$. Assume $\mathrm{kod}(Y) = 0$. Then Y has to be minimal. But then Y would be a $K3$-surface, which contradicts the fact that it admits a genus two fibration. So $\mathrm{kod}(Y) \geq 1$. Finally it follows from Proposition 4.1 in [78] p. 60 that if Y is of general type and has $K_Y^2 = 0$, then $p_g(Y) = 2$. Thus $\mathrm{kod}(Y) = 1$ and then Y is minimal by the Enriques-Kodaira classification.

□

Remark 3.34. In case $s = 0$ and the monodromy is *simple*, the singular fibres of $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ are all irreducible with a single node. Thus the only critical points of $\pi_{|B} : B \longrightarrow \mathbb{P}^1$ are smooth critical points. Therefore the branch divisor B of the morphism $\phi : Y \xrightarrow{2:1} \mathbb{F}_1$ is smooth. So in this case $\overline{B}$ is a smooth sextic in $\mathbb{P}^2$, the double cover of $\mathbb{P}^2$ branched along $\overline{B}$ is a $K3$-surface S, $\theta : S \longrightarrow \mathbb{P}^2$, and Y is S blown up in two points.

Remark 3.35. Markushevich goes in [42] the inverse way. He considers genus two fibrations $\mathfrak{f} : \mathcal{C} \longrightarrow \mathbb{P}^2$ and tries to construct a Lagrangian fibration as a relative compactification of the relative Jacobian of $\mathcal{C}/\mathbb{P}^2$. From $\mathcal{C}/\mathbb{P}^2$ he constructs a variety P together with a fibration $f : P \longrightarrow \mathbb{P}^2$. He proves that if f is a Lagrangian fibration, then one can assume that $\mathfrak{f} : \mathcal{C} \longrightarrow \mathbb{P}^2$ arises as a double cover of the projectified tangent bundle $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$,

$$\phi : \mathcal{C} \xrightarrow{2:1} \mathbb{P}(\mathcal{T}_{\mathbb{P}^2}).$$

Further he proves that if $f : P \longrightarrow \mathbb{P}^2$ is Lagrangian, then P is birational to $S^{[2]}$, where S is a $K3$-surface that is a double cover of $\mathbb{P}^{2*}$,

$$\theta : S \longrightarrow \mathbb{P}^{2*}$$

and that $\mathcal{C}/\mathbb{P}^2$ is the linear system $\{\theta^{-1}(L)\}_{L \in \mathbb{P}^2}$.

However there is one thing to note. Markushevichs definition of a family of hyperelliptic curves (Definition 4 in [42]) does not allow fibres that represent points in D_1. A curve consisting in two elliptic curves intersecting in one point has a hyperelliptic involution. But the quotient in that case consists in two copies of $\mathbb{P}^1$ that intersect in one point, instead of a single $\mathbb{P}^1$. Both in [42] and [44]

Markushevich assumes that the family of genus two curves be given locally by an equation

$$y^2 - P_6(x, u, v) = 0$$

where P_6 is a polynomial of degree six in x whose coefficients depend on local parameters u, v on $\mathbb{P}^2$. Equivalently he assumes the family to be given by a finite morphism $\mathcal{C}/\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}$ of degree 2, where $\mathbb{P}$ is a rank two vector bundle on $\mathbb{P}^2$. But in order to have a genus two fibre with a separating node the map ϕ cannot be a morphism but must be rational with indeterminacy locus contained in the fibres with separating nodes, see for example [15] and the proof of Proposition 3.37. In other words Markushevich implicitly assumes throughout that $s = 0$.$\cdot$

The $K3$-surface S he obtains is the $K3$-surface of Remark 3.34. In case the bundle $\mathbb{P}$ is indeed $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ it is easy to see that S as we defined it does not depend on the line l.

The projectified tangent bundle of $\mathbb{P}^2$ is the incidence variety $Z \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$. Let $\pi_1 : Z \longrightarrow \mathbb{P}^2$ and $\pi_2 : Z \longrightarrow \mathbb{P}^{2*}$ be the two projections. As before we denote the point in $\mathbb{P}^{2*}$ that corresponds to a line l in $\mathbb{P}^2$ by l^* and say that l^* is dual to l and vice-versa. Let l be a line in $\mathbb{P}^2$ and $l^* \in \mathbb{P}^{2*}$ the point dual to l. Then

$$\begin{aligned} Z_l &= \{(x, y) \in \mathbb{P}^2 \times \mathbb{P}^{2*} \, | x \in l, \, \sum_{i=0}^{2} x_i y_i = 0\} \\ &= \{(x, y) \in l \times \mathbb{P}^{2*} \, | \sum_{i=0}^{2} x_i y_i = 0\}. \end{aligned}$$

So the map

$$p_l := \pi_{2|Z_l} : Z_l \longrightarrow \mathbb{P}^{2*}$$

is the blow up of $\mathbb{P}^{2*}$ in l^*. The branch locus of ϕ is a divisor B on Z. As $Z = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ its Picard group is $\mathrm{Pic}(Z) = \mathbb{Z}^2$ and consists in elements $\mathcal{O}_Z(a, b) := \pi_1^* \mathcal{O}_{\mathbb{P}^2}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{2*}}(b)$. As $s = 0$, the divisor B has bidegree $(0, 6)$.

Now let l be a general line and $B_l := B_{|l}$. Since $s = 0$ the projection $\pi_{1|B_l}$: $B_l \longrightarrow l$ has only smooth critical points. Therefore B_l is smooth. Furthermore by equation (39) $s = 0$ implies that B_l does not intersect the exceptional divisor of p_l. Consequently its image $\overline{B}_l := p_l(B_l)$ is a smooth sextic in $\mathbb{P}^{2*}$. We show that this sextic does not depend on the line l. Thus let l_1 be another general line in $\mathbb{P}^2$ and b be a point on $\overline{B}_l$. The two lines $\overline{bl^*}$, $\overline{bl_1^*}$ in $\mathbb{P}^{2*}$ are dual to points $x \in l$ and

$x_1 \in l_1$. Consequently b is dual to the line $g := \overline{xx_1}$ and $b = x^* \cap x_1^*$. Now

$$x_1^* \cap \overline{B}_{l_1} = p_{l_1}(Z_{x_1} \cap B) = \mathrm{pr}_2(Z_{x_1} \cap B) = p_g(Z_{x_1} \cap B) = x_1^* \cap \overline{B}_g$$

and analogously

$$x^* \cap \overline{B}_l = x^* \cap \overline{B}_g.$$

As $b = (x^* \cap \overline{B}_g) \cap (x_1^* \cap \overline{B}_g)$, it follows that $b \in p_{l_1}(B_{l_1})$. As b and l_1 were arbitrary, we get $\overline{B}_l \subset \overline{B}_{l_1}$ for all lines l_1 in $\mathbb{P}^2$. From this we deduce $\overline{B}_l = \overline{B}_{l_1}$. Thus there is a smooth sextic $\overline{B}$ in $\mathbb{P}^{2*}$. The fibre $\mathcal{C}_x$ over a point $x \in \mathbb{P}^2$ is the double cover of the line x^* branched in $x^* \cap \overline{B}$. Thus $\mathcal{C}$ is a linear system on the $K3$-surface S, that is the double cover of $\mathbb{P}^{2*}$ branched along $\overline{B}$.

Remark 3.36. Using the fact that the family $\mathcal{C}$ is a linear system on a $K3$-surface Markushevich constructs a birational map from $S^{[2]}$ to P. Contrary to the above in case $s \neq 0$ the branch locus B_l on $\mathbb{F}_1 = Bl_{l^*}(\mathbb{P}^{2*})$ is not the transform of a fixed divisor on $\mathbb{P}^{2*}$. Equation (39) implies that s is the intersection number of B_l with the exceptional divisor. So $s \neq 0$ implies that $\overline{B}_l := p_l(B_l)$ has an s-fold point in l^*. Thus for $l \neq l'$, $\overline{B}_l := p_l(B_l) \neq p_l(B_{l'}) =: \overline{B}_{l'}$. The important point in the proof that P is birational to $S^{[2]}$ is the existence of a fixed $K3$-surface S that is guaranteed by the fact that $s = 0$. For $s \neq 0$ this fails in two respects. Firstly Y is not birational to a $K3$-surface, and secondly it is not clear that the family of curves is a linear system on a fixed surface.

There are no known examples of Lagrangian fibrations with $s > 0$. Example 3.24 has $s = 0$. It is therefore natural to ask whether there exist Lagrangian fibrations with $s > 0$. A necessary condition for this is the existence of the corresponding genus two fibration. Therefore we ask whether appropriate genus two fibrations $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ exist.

3.7. **Example of a surface with** $(s = 2)$**.** A genus two fibration that comes from a Lagrangian fibration has base $\mathbb{P}^1$, $\mathfrak{f}_* K_{Y/\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ and $n + 2s = 30$. We ask whether there exist such genus two fibrations with $s > 0$. This turns out to be the case. We construct an example with $n = 26$ and $s = 2$.

Proposition 3.37. *There exists a genus two fibration* $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ *with* $\mathfrak{f}_* K_{Y/\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ *and only nodal singular fibres that has* 26 *non-separating nodes and* 2 *fibres with a single separating node each.*

We construct such a surface Y together with a rational map

$$\phi : Y - - > \mathbb{F}_1$$

that is generically of degree two. Let u_1, u_2 be two different points in $\mathbb{P}^2$ and L_1, L_2 be two lines in $\mathbb{P}^2$ passing through u_1 and u_2 respectively, but not through both. Consider the family $\mathcal{Q}$ of plane quadrics that are tangent to L_i in u_i for $i = 1, 2$. As the family of plane quadrics is 5-dimensional, the family $\mathcal{Q}$ is a pencil. Let Q_j for $j = 0, 1, 2$ be three general elements of $\mathcal{Q}$. The curve $Q_0 + Q_1 + Q_2$ has then simple infinitely close triple points in u_i, $i = 1, 2$ and is otherwise smooth. Choose a point p_∞ that does not lie on any of the Q_j's nor on any of the L_i's and choose a line $l \subset \mathbb{P}^2$ in general position with respect to p_∞, the Q_j and the L_i. Blowing up p_∞ yields $\mathbb{F}^1$ with a projection $\pi : \mathbb{F}_1 \longrightarrow l$. The strict transform under this blow up of the curve $Q_0 + Q_1 + Q_2$ (which we denote by the same letters) is a divisor on $\mathbb{F}^1$ that is smooth apart from two simple infinitely close triple points (that we likewise denote by u_1 and u_2). Consider the following divisor on $\mathbb{F}^1$:

$$B_0 := Q_0 + Q_1 + Q_2 + M_1 + M_2,$$

where $M_i, i = 1, 2$ are two different fibres of $\mathbb{F}^1 \longrightarrow \mathbb{P}^1$, that intersect $Q_0 + Q_1 + Q_2$ transversally. This divisor has two simple infinitely close triple points in different fibres of $\pi : \mathbb{F}_1 \longrightarrow \mathbb{P}^1$ and apart from that only simple nodes.

Lemma 3.38. *The divisor B_0 can be deformed into a divisor B of bidegree $(2, 6)$ on $\mathbb{F}_1$ that satisfies the following conditions. It has two simple infinitely close triple points u_1, u_2 in different fibres and the critical points of the projection $\pi_{|B} : B \longrightarrow \mathbb{P}^1$ other than u_1, u_2 are simple smooth critical points.*

Proof: Embedded into $\mathbb{P}^1 \times \mathbb{P}^2$,

$$\mathbb{F}_1 = \{([x_0, x_2], [y_0, y_1, y_2]) \in \mathbb{P}^1 \times \mathbb{P}^2 |\ x_0 y_0 + x_2 y_2 = 0\}.$$

Let $\pi_1 := \mathrm{pr}_{1|\mathbb{F}_1} : \mathbb{F}_1 \longrightarrow \mathbb{P}^1$ and $\pi_2 := \mathrm{pr}_{2|\mathbb{F}_1} : \mathbb{F}_1 \longrightarrow \mathbb{P}^2$. We denote points in $\mathbb{P}^1$ by $X = [x_0, x_2]$ and points in $\mathbb{P}^2$ by $Y = [y_0, y_1, y_2]$. For points in $\mathbb{F}_1$, i.e. elements (X, Y) of $\mathbb{P}^1 \times \mathbb{P}^2$ that satisfy the equation $x_0 y_0 + x_2 y_2 = 0$, we use the notation $(\overline{x}, \overline{y})$. Let $F_0(Y), F_1(Y) \in \mathbb{C}[y_0, y_1, y_2]$ be equations for Q_0, Q_1. An equation for Q_2 can then be written as

$$\lambda_0 F_0(Y) + \lambda_1 F_1(Y).$$

Let $Q(X) \in \mathbb{C}[x_0, x_2]$ be a homogeneous polynomial of degree two. Let

$$f_0(X, Y) := Q(X)F_0(Y)F_1(Y)(\lambda_0 F_0(Y) + \lambda_1 F_1(Y)).$$

Then $f_0(\overline{x}, \overline{y}) = 0$ is an equation for B_0. Let now $G(\overline{x}, W_0, W_1)$ be a homogeneous polynomial of degree 3 in (W_0, W_1) with coefficients homogeneous polynomials of degree 2 in $\overline{x}$. Let

$$f(X, Y) = G(X, F_0(Y), F_1(Y)),$$

i.e.

$$f(X,Y) = a_3(X)F_0(Y)^3 + a_2(X)F_0(Y)^2F_1(Y) + a_1(X)F_0(Y)F_1(Y)^2 + a_0(X)F_1(Y)^3$$

where a_i are homogeneous polynomials of degree two in X. Then the equations $(f(\overline{x}, \overline{y}) = 0)$ define a linear system E on $\mathbb{F}_1$ that is eleven dimensional and of bidegree $(2, 6)$. The base locus of E consists in the two points u_1, u_2. Clearly the two points lie in the base locus. Conversely let $(\overline{x}', \overline{y}')$ be in the base locus of E. Then

$$a_3(\overline{x}')F_0(\overline{y}')^3 + a_2(\overline{x}')F_0(\overline{y}')^2F_1(\overline{y}') + a_1(\overline{x}')F_0(\overline{y}')F_1(\overline{y}')^2 + a_0(\overline{x}')F_1(\overline{y}')^3 = 0,$$

for all quadrics $a_i, i = 0, ..., 3$. This implies $F_0(\overline{y}') = F_1(\overline{y}') = 0$. So $(\overline{x}', \overline{y}')$ is either u_1 or u_2. By Bertini's theorem the general element of E is thus smooth away from u_1 and u_2. A general element of E has three smooth branches in the points u_1, u_2. Let $(\overline{x}', \overline{y}') = u_i$. In the point $\overline{x}'$ the homogeneous cubic G is

$$G(\overline{x}', W_0, W_1) = \prod_{k=1}^{3}(\mu'_k W_0 - \nu'_k W_1)$$

for $\nu'_k, \mu'_k \in \mathbb{C}$. For general $a_0, ..., a_3$ the ν'_k, μ'_k define points $[\nu'_k, \mu'_k] \in \mathbb{P}^1$ that are pairwise distinct. Thus each of the

$$(\mu'_k F_0(\overline{y}) - \nu'_k F_1(\overline{y}))$$

for $k = 1, 2, 3$ has two simple roots, one of them being $\overline{y}'$. Locally in $\mathbb{P}^1$ around $\overline{x}'$

$$G(\overline{x}, W_0, W_1) = \prod_{k=1}^{3}(\mu_k W_0 - \nu_k W_1)$$

with the μ_k, ν_k for $k = 1, 2, 3$ holomorphic functions in $\overline{x}$, such that the $[\nu_k, \mu_k] \in \mathbb{P}^1$ are pairwise distinct. It follows that $(f(\overline{x}, \overline{y}) = 0)$ has locally in $\mathbb{P}^1$ around $\overline{x}'$ six smooth branches, three of which meet in u_i.

Furthermore the three branches meeting in u_i are tangent to each other. Assume that u_1 and u_2 lie in $\{x_0 \neq 0, y_2 \neq 0\} \subset \mathbb{P}^1 \times \mathbb{P}^2$ and let (v, y) with $v = \frac{y_0}{y_2}, y = \frac{y_1}{y_2}$ be affine coordinates of $\mathbb{P}^2$ and $x = \frac{x_2}{x_0}$ an affine coordinate on $\mathbb{P}^1$. Then $\mathbb{F}_1$ is given by $(x + v = 0)$ and (x, y) are affine coordinates on $\mathbb{F}_1$ with $(x, y) \leftrightarrow ([1, x], [-x, y, 1])$. The projection is given by $(x, y) \mapsto x$. Let

$$s \mapsto (v(s), y(s))$$

be a parametrisation of L_i in the affine coordinates (v, y) of $\mathbb{P}^2$. Then

$$s \mapsto (-v(s), y(s))$$

is a parametrisation of $\pi_2^{-1}(L_i)$ in the coordinates (x, y) of $\mathbb{F}_1$. Let $x(s) := -v(s))$ and write $F_j(x, y)$ and $F_j(v, y)$ for F_j in the coordinates (v, y) of $\mathbb{P}^2$ and (x, y) of $\mathbb{F}_1$ respectively. As $F_j(v(s), y(s))$ for $j = 1, 2$ vanishes to order 2 in $s = 0$, each of the

$$(\mu_k(x(s))F_0(x(s), y(s)) - \nu_k(x(s))F_1(x(s), y(s)))$$

vanishes to order 2 in $s = 0$. Therefore the three branches meeting in u_i are tangent to L_i in u_i.

In order to prove the remaining condition on the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^1$, we first deform $Q_2 + M_1 + M_2$ and leave $Q_0 + Q_1$ untouched. We do this by deforming the equation $Q(\overline{x})(\lambda_0 F_0(\overline{y}) + \lambda_1 F_1(\overline{y})) = 0$ to

$$g(\overline{x}, \overline{y}) = A_0(\overline{x})F_0(\overline{y}) + A_1(\overline{x})F_1(\overline{y}),$$

where $A_j, j = 1, 2$ are homogeneous polynomials of degree 2 in $\mathbb{P}^1$. Denote the corresponding divisor by B_1. The B_1's form a 5-dimensional linear system E_1 of bidegree $(2, 2)$ on $\mathbb{F}_1$. By Bertini's theorem the general element of E_1 is smooth away from the two base points u_1, u_2. A simple calculation in affine coordinates shows that the general element of E_1 is smooth also in u_1 and u_2.

Take again the affine coordinates (x, y) on $\mathbb{F}_1$. Write $F_j(x, y)$ for F_j in these coordinates and $A_i(x)$ for A_i in the coordinate x on $\mathbb{P}^1$ and put

$$g(x, y) := A_0(x)F_0(x, y) + A_1(x)F_1(x, y).$$

Then $B_1 = (g = 0)$. The conditions for a smooth critical point of $\pi_{1|B_1} : B_1 \longrightarrow \mathbb{P}^1$ are

$$\begin{aligned} g &= 0 \\ \frac{\partial g}{\partial y} &= 0. \end{aligned}$$

The discriminant $\mathrm{Discr}(g, y)$ of g with respect to y is a polynomial of degree 6 in x. As the bidegree of B_1 is $(2,2)$ this implies that for a smooth element of E_1 the projection $\pi_{1|B_1} : B_1 \longrightarrow \mathbb{P}^1$ has 6 distinct smooth critical points. And for general quadrics A_j, the four points $A_0(x) = 0, A_1(x) = 0$ are not among the six singular values.

For B_1 a general element of E_1 consider now the divisor

$$Q_0 + Q_1 + B_1.$$

It has two simple infinitely close triple points in different fibres. The remaining critical points of $\pi_{1|Q_0+Q_1+B_1} : Q_0 + Q_1 + B_1 \longrightarrow \mathbb{P}^1$ are six smooth critical points coming from B_1, four smooth critical points coming from Q_0 and Q_1 and transversal intersections of B_1 with $Q_0 + Q_1$. The intersection number of B_1 with $Q_0 + Q_1$ is

$$(2F + 2H) \cdot 2(2H) = 16.$$

Suppose $(x, y) \in B_1 \cap Q_0$. Then $A_1(x)F_1(y) = 0$. So either $(x, y) \in \{u_1, u_2\}$ or $A_1(x) = 0$. We can assume that the two points $A_1(x) = 0$ are not singular values of $\pi_{|Q_0} : Q_0 \longrightarrow \mathbb{P}^1$ or $\pi_{|Q_1} : Q_1 \longrightarrow \mathbb{P}^1$. Thus $B_1 \cap Q_0$ consists of u_1, u_2 together with two pairs of simple nodes over the two roots of A_1. And analogously for $B_1 \cap Q_1$. The critical locus of

$$\pi_{1\,|Q_0+Q_1+B_1} : Q_0 + Q_1 + B_1 \longrightarrow \mathbb{P}^1$$

consists therefore in the six smooth critical points of B_1, the two smooth critical points from each of Q_0 and Q_1, the two simple infinitely close triple points u_1, u_2 and four pairs of simple nodes over the roots of A_0, A_1. A deformation of $Q_0 + Q_1 + B_1$ that smoothes all points except for u_1, u_2 deforms each node in two simple smooth critical points. Thus for a general element B of E the projection $\pi_{1\,|B} : B \longrightarrow \mathbb{P}^1$ has

$$6 + 4 + 2 \cdot 8 = 26$$

simple smooth critical points.

□

Proof of Proposition 3.37: For a general element B of E, the the infinitely close triple points lie in fibres that contain no other critical point of $\pi_{|B} : B \longrightarrow \mathbb{P}^1$. As to the 26 smooth critical points there are no more than two of them in each fibre. In fact at least 10 of the singular fibres contain only one smooth critical point.

From such a divisor B we can construct a smooth surface Y with a rational map $\phi : Y - - > \mathbb{F}_1$ generically of degree 2 that is branched along B. The construction is as follows. We first blow up the two infinitely close triple points s_1, s_2 of B

$$\sigma : \widetilde{\mathbb{F}_1} \longrightarrow \mathbb{F}_1.$$

Let E_1, E_2 be the two exceptional divisors of σ. The strict transform $B_{\widetilde{\mathbb{F}_1}}$ of B has two simple triple points $\widetilde{s}_1, \widetilde{s}_2$. Let

$$\tau : \widetilde{\widetilde{\mathbb{F}_1}} \longrightarrow \widetilde{\mathbb{F}_1}$$

be the blow up of $\widetilde{s}_1, \widetilde{s}_2$ and denote by Γ_1, Γ_2 the strict transforms of E_1, E_2 under τ. These are (-2)-curves. Let Γ_1', Γ_2' be the two exceptional divisors of τ. The strict transform $B_{\widetilde{\widetilde{\mathbb{F}_1}}}$ of $B_{\widetilde{\mathbb{F}_1}}$ is now smooth. Denote by $\widetilde{B}$ the union of $B_{\widetilde{\widetilde{\mathbb{F}_1}}}$ with Γ_1 and Γ_2. Consider the total transform

$$\begin{aligned} \tau^*\sigma^*B &= \tau^*(B_{\widetilde{\mathbb{F}_1}} + 3\sum_{i=1}^{2} E_i) \\ &= B_{\widetilde{\widetilde{\mathbb{F}_1}}} + 3\sum_{i=1}^{2}\Gamma_i' + 3\sum_{i=1}^{2}(\Gamma_i + \Gamma_i') \\ &= B_{\widetilde{\widetilde{\mathbb{F}_1}}} + \sum_{i=1}^{2}\Gamma_i + 2\sum_{i=1}^{2}(3\Gamma_i' + \Gamma_i). \end{aligned}$$

The divisor B is linearly equivalent to $6H + 2F$ and 2-divisible in $\mathrm{Pic}(\mathbb{F}_1)$. Now the last equality implies that $B_{\widetilde{\widetilde{\mathbb{F}_1}}} + \sum_{i=1}^{2}\Gamma_i$ is 2-divisible in $\mathrm{Pic}(\widetilde{\widetilde{\mathbb{F}_1}})$. We can thus construct the double cover of $\widetilde{\widetilde{\mathbb{F}_1}}$ branched along $B_{\widetilde{\widetilde{\mathbb{F}_1}}} + \sum_i \Gamma_i$. This is a smooth surface $\widetilde{Y}$

$$\delta : \widetilde{Y} \xrightarrow{2:1} \widetilde{\widetilde{\mathbb{F}_1}}.$$

As the Γ_i are (-2)-curves, their reduced pullbacks $\mathcal{E}_i$ under π are (-1)-curves in $\widetilde{Y}$. Let

$$\epsilon : \widetilde{Y} \longrightarrow Y$$

be the simultaneous contraction of these. This gives a commutative diagram

$$\begin{tikzcd} Y \arrow[r, dashed, "\phi"] & \mathbb{F}_1 \\ \widetilde{Y} \arrow[u, "\epsilon"] \arrow[r, "\delta"] & \widetilde{\widetilde{\mathbb{F}_1}}, \arrow[u, "\tau\circ\sigma"'] \end{tikzcd}$$

where ϕ is a rational map with indeterminacy locus the fundamental points of ϵ. The induced fibration $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ is a genus two fibration with non-separating nodes in fibres where $\pi_{|B} : B \longrightarrow \mathbb{P}^1$ has smooth critical points. There are 26 of these. On the other hand the fibre of $\mathfrak{f}$ over a point $\pi(s_i)$ consists in two elliptic curves that intersect transversally in one point. All other fibres are smooth.

It remains to show that $\mathfrak{f}_* K_{Y/\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. By a formula of Horikawa [78] p. 13,

$$K_Y^2 = 2\chi(\mathcal{O}_Y) - 6 + h,$$

where the number h depends on the singular fibres of $\mathfrak{f}$. In our case $h = s$. As the Euler characteristic of Y is

$$\chi_{\text{top}}(Y) = -4 + n + s = 24,$$

Noether's formula gives

$$\begin{aligned} 24 &= 12\chi(\mathcal{O}_Y) - K_Y^2 \\ &= 10\chi(\mathcal{O}_Y) + 4. \end{aligned}$$

Thus $\chi(\mathcal{O}_Y) = 2$, $\deg(\mathfrak{f}_* K_{Y/\mathbb{P}^1}) = 3$ and $K_Y^2 = 0$. Theorem 2.2 in [78] implies $K_Y^2 \geq 3e - 5$ and thereby $e = 1$. From this we conclude $\mathbb{P}((\mathfrak{f}_* K_{Y/\mathbb{P}^1})^*) \simeq \mathbb{F}_1$. Therefore

$$(\mathfrak{f}_* K_{Y/\mathbb{P}^1})^* \simeq (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{L}$$

for a line bundle $\mathcal{L}$ on $\mathbb{P}^1$. $\deg(\mathfrak{f}_* K_{Y/\mathbb{P}^1}) = 3$ implies $\deg(\mathcal{L}) = -2$. So $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and

$$\mathfrak{f}_* K_{Y/\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2).$$

□

We can say more on the distribution of the critical points.

Proposition 3.39. *For the general element of the linear system E the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^1$ has exactly 20 distinct critical values. Two from the triple points, 10 from fibres with one simple smooth critical point each and 8 from fibres with two simple smooth critical points each.*

Proof: Let $b_1 := \pi_1(u_1)$, $b_2 := \pi_1(u_2)$ and $B = (F_B = 0)$ be an element of E,

$$\begin{aligned} F_B(\overline{x}, \overline{y}) &= G(\overline{x}, F_0, F_1) \\ &= A_3 F_0^3 + A_0 F_0^2 F_1 + A_1 F_0 F_1^2 + A_2 F_1^3. \end{aligned}$$

For $b \in \mathbb{P}^1$ let y be homogeneous coordinates on the fibre $\mathbb{F}_{1b}$ and write $F_{0b}(y), F_{1b}(y)$ for the quadrics $F_{0|\mathbb{F}_{1b}}, F_{1|\mathbb{F}_{1b}}$ in these coordinates. For $b \in \mathbb{P}^1$, $b \neq b_1, b_2$ let $F_{B,b}(y) := F_B(b, y)$, i.e.

$$F_{B,b}(y) = (\mu_{1b}F_{0b}(y) - \nu_{1b}F_{1b}(y))(\mu_{2b}F_{0b}(y) - \nu_{2b}F_{1b}(y))(\mu_{3b}F_{0b}(y) - \nu_{3b}F_{1b}(y)).$$

By the above proof it suffices to show that for a general element B of E there are at least 8 points b such that $F_{B,b}(y)$ has two simple and two twofold zeros. Assume that two of the $[\nu_{kb}, \mu_{kb}]$ are the same, i.e.

$$F_{B,b}(y) = (\mu_{1b}F_{0b}(y) - \nu_{1b}F_{1b}(y))^2(\mu_{2b}F_{0b}(y) - \nu_{2b}F_{1b}(y)). \tag{40}$$

In that case the roots of $F_{B,b} = 0$ on $\mathbb{F}_{1b}$ are two simple and two twofold zeros. If we set $A_3(b) = q, A_0(b) = r, A_1(b) = s, A_2(b) = t$, then (40) implies

$$r^2s^2 - 4r^3t - 4qs^3 + 18qrst - 27q^2t^2 = 0.$$

This is the discriminant of G with respect to (W_0, W_1) and defines an irreducible quartic D in $\mathbb{P}^3$. The A_i on the other hand define a curve

$$\begin{array}{ccc} \mathbb{P}^1 & \stackrel{A}{\longrightarrow} & \mathbb{P}^3 \\ b & \mapsto & [A_3(b), A_0(b), A_1(b), A_2(b)] \end{array}$$

of degree two in $\mathbb{P}^3$. For general A_i, this curve will intersect D in exactly 8 points. So for a general choice of the A_i the projection will have eight singular values that stem each from a pair of smooth critical points.

□

The general divisor of bidegree $(2, 6)$ on $\mathbb{F}_1$ which has two infinitely close triple points as singularities however does not lie in the linear system E.

Let B be a divisor of bidegree $(2, 6)$ on $\mathbb{F}_1$ with two infinitely close triple points as its only singularities. Let $\sigma : \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ be the blow down of the exceptional divisor S_∞. As $B \cdot S_\infty = 2$, the curve $O := \sigma_* B$ satisfies

$$\begin{aligned} O \cdot \mathbb{P}^1 &= \sigma^* O \cdot \sigma^* \mathbb{P}^1 \\ &= (B + 2S_\infty) \cdot H \\ &= 8. \end{aligned}$$

Thus O is an octic in $\mathbb{P}^2$ with two infinitely close triple points and a node in the fundamental point of σ. Conversely from an octic O in $\mathbb{P}^2$ with these singularities one can produce a divisor on $\mathbb{F}_1$ with the above mentioned properties by blowing up $\mathbb{P}^2$ in the node of O. There is a 19-dimensional family of such octics on $\mathbb{P}^2$. In homogeneous coordinates $[X, Y, Z]$ on $\mathbb{P}^2$, let $u_1 = [0, 0, 1]$, $u_2 = [1, 0, 0]$ and

$$\begin{aligned} L_1 &: X + Y = 0 \\ L_2 &: Z + Y = 0. \end{aligned}$$

In affine coordinates $x = \frac{X}{Z}, y = \frac{Y}{Z}$ on $U_2 = \{Z \neq 0\}$ the condition for an octic O given by $F(X, Y, Z) = 0$ to have an infinitely close triple point in u_1 with tangent L_1 is that $F_2 := F(x, y, 1)$ has the form

$$\begin{aligned} F_2(x, y) &= \lambda'(x + y)^3 + (x + y)h_3(x, y) + (x + y)h_4(x, y) \\ &\quad +(x + y)h_5(x, y) + \lambda x^6 + h_7(x, y) + h_8(x, y), \end{aligned}$$

where h_k is a homogeneous polynomial of degree k and $\lambda, \lambda' \neq 0$. Analogously in affine coordinates $y = \frac{Y}{X}, z = \frac{Z}{X}$ on $U_0 = \{X \neq 0\}$ the condition that O has an infinitely close triple point in u_2 with tangent L_2 is that $F_0 := F(1, y, z)$ has the form

$$\begin{aligned} F_0(z, y) &= \mu'(z + y)^3 + (z + y)g_3(z, y) + (z + y)g_4(z, y) \\ &\quad +(z + y)g_5(z, y) + \mu z^6 + g_7(z, y) + g_8(z, y), \end{aligned}$$

where g_k is a homogeneous polynomial of degree k and $\mu, \mu' \neq 0$. Thus

$$\begin{aligned} F(X, Y, Z) &= \lambda'(X + Y)^3 Z^5 + (X + Y)h_3(X, Y)Z^4 + (X + Y)h_4(X, Y)Z^3 \\ &\quad +(X + Y)h_5(X, Y)Z^2 + \lambda X^6 Z^2 + h_7(X, Y)Z + h_8(X, Y). \end{aligned}$$

Let $h_k = \sum_{i=0}^{k} h_{ki} x^{k-i} y^i$ and $g_k = \sum_{i=0}^{k} g_{ki} z^{k-i} y^i$. Then

$$\begin{aligned}
F_0(z,y) &= \lambda'(1+y)^3 z^5 + (1+y)h_3(1,y)z^4 + (1+y)h_4(1,y)z^3 \\
&\quad +(1+y)h_5(1,y)z^2 + \lambda z^2 + h_7(1,y)z + h_8(1,y) \\
&= \lambda'(z^5 + 3yz^5 + 3y^2z^5 + y^3z^5) \\
&\quad + yz^4 \sum_{i=0}^{3} h_{3i} y^i + z^4 \sum_{i=0}^{3} h_{3i} y^i \\
&\quad + yz^3 \sum_{i=0}^{4} h_{4i} y^i + z^3 \sum_{i=0}^{4} h_{4i} y^i \\
&\quad + yz^2 \sum_{i=0}^{5} h_{5i} y^i + z^2 \sum_{i=0}^{5} h_{5i} y^i \\
&\quad + \lambda z^2 + z \sum_{i=0}^{7} h_{7i} y^i + \sum_{i=0}^{8} h_{8i} y^i.
\end{aligned}$$

The condition that O has a double point in $[0,1,0]$ implies that

$$h_{88} = h_{87} = h_{77} = 0.$$

The condition that O has an infinitely close triple point in $[1,0,0]$ yields

$$\begin{aligned}
F_0(z,y) &= \mu' z^3 + 3\mu' z^2 y + 3\mu' z y^2 + \mu' y^3 \\
&\quad + y \sum_{i=0}^{3} g_{3i} z^{3-i} y^i + z \sum_{i=0}^{3} g_{3i} z^{3-i} y^i \\
&\quad + y \sum_{i=0}^{4} g_{4i} z^{4-i} y^i + z \sum_{i=0}^{4} g_{4i} z^{4-i} y^i \\
&\quad + y \sum_{i=0}^{5} g_{5i} z^{5-i} y^i + z \sum_{i=0}^{5} g_{5i} z^{5-i} y^i \\
&\quad + \mu z^6 + \sum_{i=0}^{6} g_{7i} z^{7-i} y^i + \sum_{i=0}^{6} g_{8i} z^{8-i} y^i.
\end{aligned}$$

Compairing the coefficients yields 11 additional conditions on the h_{ki}, λ' and λ:

$$\begin{aligned}
h_{80} &= h_{81} = h_{82} = h_{70} = h_{71} = 0 \\
h_{50} &= -\lambda \\
h_{83} &= h_{40} \\
h_{72} &= 3h_{40} \\
h_{51} &= 3h_{40} + \lambda \\
h_{30} &- h_{40} - h_{41} + h_{51} + h_{52} - h_{73} + h_{84} = 0 \\
h_{30} &+ h_{31} - h_{41} - h_{42} + h_{52} + h_{53} - h_{74} + h_{85} - \lambda' = 0.
\end{aligned}$$

As these conditions are all linear, these octics form a linear system $\mathfrak{O}$. This linear system has dimension

$$33 - 14 = 19$$

and by Bertini's theorem the general element is smooth away from the points $[0,0,1],[1,0,0],[0,1,0]$. Recall that the dimension of the linear system E was 11. We used `SINGULAR 3.0.2.`[23] to generate random numbers as values for the h_{ki}, λ' and λ. For random values so obtained we studied the corresponding curve B in $\mathbb{F}_1$ and the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^1$. Again using `SINGULAR 3.0.2.`we found that the smooth critical points of the projection were all simple and that the projection did not have more than one smooth critical point in a fibre. Thus

Proposition 3.40. *There exist divisors B of bidegree $(2,6)$ on $\mathbb{F}_1$ with two infinitely close triple points as the only singularities and such that the smooth critical points of the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^1$ are 26 simple smooth critical points that lie 26 fibres which are different from the ones containing the triple points.*

□

As in the proof of Proposition 3.37 this implies:

Corollary 3.41. *There exists a genus two fibration $\mathfrak{f} : Y \longrightarrow \mathbb{P}^1$ with $\mathfrak{f}_* K_{Y/\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ and only nodal singular fibres that has 26 irreducible fibres with a single non-separating node each and 2 reducible fibres with a single separating node each.*

□

3.8. **Construction of an abelian fibration.** Inspired by the construction of example 3.7, we try to construct a Lagrangian fibration over $\mathbb{P}^2$ such that $\deg(\Delta) = 26$.

Let Z be the incidence variety in $\mathbb{P}^2 \times \mathbb{P}^{2*}$ and $\pi_1 : Z \longrightarrow \mathbb{P}^2,\ \pi_2 : Z \longrightarrow \mathbb{P}^{2*}$ the projection to the first and second component respectively. As in example 3.7 we take two points $u_1, u_2 \in \mathbb{P}^{2*}$ and L_1, L_2 two different lines in $\mathbb{P}^{2*}$ passing through u_1 and u_2 respectively but not through both. Let $\mathcal{Q}$ be the pencil of plane quadrics in $\mathbb{P}^{2*}$ that are tangent to L_i in u_i, for $i = 1, 2$ and let Q_0, Q_1, Q_2 be three general elements of $\mathcal{Q}$. Furthermore let Q be a smooth quadric in $\mathbb{P}^2$ that is in general position with respect to the quadrics Q_j^* dual to $Q_j,\ j = 0, 1, 2$. Consider the divisor

$$B_0 := \pi_1^* Q + \pi_2^* Q_0 + \pi_2^* Q_1 + \pi_2^* Q_2$$

on Z. It has bidegree $(2, 6)$. We want to deform this divisor into a divisor B on Z in such a way that we can construct a smooth threefold Y with a rational map $\phi : Y - - > Z$ that is generically of degree 2, branched along B and such that over a general line l in $\mathbb{P}^2$ the genus two fibration $Y_{|l}$ has two fibres with a separating node and contains 26 separating nodes. The idea is then to construct from this a an abelian fibration over $\mathbb{P}^2$ as the relative compactified Jacobian of this genus two fibration. The discriminant locus of the abelian fibration has then degree 26.

Let us fix some notation. By $X = [x_0, x_1, x_2]$ and $Y = [y_0, y_1, y_2]$ we denote elements of $\mathbb{P}^2$ and $\mathbb{P}^{2*}$ respectively. For points in Z we use the notation $(\overline{x}, \overline{y})$. Let

$$S_i := \pi_2^{-1}(u_i)$$

for $i = 1, 2$. These are lines in Z

$$S_i = \{(X, u_i) \in \mathbb{P}^2 \times \mathbb{P}^{2*} | X \cdot u_i = 0\} = s_i \times \{u_i\},$$

where s_i denotes the line in $\mathbb{P}^2$ dual to $u_i \in \mathbb{P}^{2*}$. Furthermore denote by q the point $s_1 \cap s_2$, by t_i the point in $\mathbb{P}^2$ dual to L_i and let

$$H_i := \pi_2^{-1}(L_i)$$

for $i = 1, 2$. Then

$$H_i = \{(X, Y) \in \mathbb{P}^2 \times \mathbb{P}^{2*} \ |Y \in L_i,\ X \cdot Y = 0\}.$$

Let $F_0(Y), F_1(Y) \in \mathbb{C}[y_0, y_1, y_2]$ be two homogeneous polynomials such that $Q_j = (F_j(Y) = 0)$ for $j = 0, 1$. Then there are $\lambda_0, \lambda_1 \in \mathbb{C}$ such that $Q_2 = (\lambda_0 F_0(Y) +$

$\lambda_1 F_1(Y) = 0)$. Let $P(X) \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial such that $P(X) = 0$ is an equation for Q. Let

$$f_0(X, Y) = P(X)F_0(Y)F_1(Y)(\lambda_0 F_0(Y) + \lambda_1 F_1(Y)).$$

Then $(f_0(\overline{x}, \overline{y}) = 0)$ is an equation for B_0. B_0 contains S_1, S_2.

Let $p = (\overline{x}', \overline{y}')$ be a general point on S_i. Then in a neighbourhood of $\overline{x}'$ in $\mathbb{P}^2$ $P(X) \neq 0$. Above this neighbourhood B_0 has six smooth branches. Three of these branches intersect along S_i. More precisely these three branches are tangent to H_i along S_i.

Now let $G(X, W_0, W_1)$ be a homogeneous polynomial of degree three in (W_0, W_1) with coefficients homogeneous polynomials of degree two in X, i.e.

$$G(X, W_0, W_1) = a_3(X)W_0^3 + a_2(X)W_0^2 W_1 + a_1(X)W_0 W_1^2 + a_0(X)W_1^3$$

with $a_j \in \mathbb{C}[x_0, x_1]$ homogeneous of degree two. Let

$$f(X, Y) = G(X, F_0(Y), F_1(Y)).$$

Then the equations $(f(\overline{x}, \overline{y}) = 0)$ define a 23-dimensional linear system E of bidegree $(2, 6)$ on Z. This is our deformation of B_0. The base locus of this linear system consists precisely in the lines S_1, S_2. By Bertini's theorem the general element of E is thus smooth away from S_1, S_2. Over a neighbourhood in $\mathbb{P}^2$ of a general point on $s_1 \cup s_2$ the general element of E is isomorphic to the disjoint union of three copies of

$$\text{disc} \times \text{disc}$$

with

$$\text{simple infinitely close triple point} \times \text{disc},$$

where disc $:= \{z \in \mathbb{C} | |z| < 1\}$. The infinitely close triple points occur along S_i.

Let B be a general element of E and denote by D the set of singular values of the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^2$. The projection has a *simple smooth critical point* along a curve $R \subset B$ if locally around a point on R there exist coordinates (z_1, z_2, z_3) on Z such that B is given by $(z_3 = z_1^2)$ and R by $(z_1 = z_3 = 0)$. In case the projection has no other critical points over the curve $\pi_{1|B}(R)$, we say that it has a single simple smooth critical point over $\pi_{1|B}(R)$.

Proposition 3.42. *The general element B of the linear system E satisfies the following conditions. It contains the lines S_1, S_2 and its only singularities occur on S_1 and S_2. Locally around a general point of S_i it is isomorphic to*

$$\text{simple infinitely close triple point} \times \text{disc}\,.$$

The critical points of $\pi_{1|B} : B \longrightarrow \mathbb{P}^2$ away from $S_1 \cup S_2$ are as follows. There is a curve Δ_1 of degree 10 in $\mathbb{P}^2$ such that over a general point of Δ_1 lies a single simple smooth critical point. Furthermore there is a curve $\Delta_{2,\mathrm{red}}$ of degree 8 such that over a general point of $\Delta_{2,\mathrm{red}}$ lie two simple smooth critical points.

Proof: We first discuss the situation that arises when we deform $\pi_1^*Q + \pi_2^*Q_2$ and leave $\pi_2^*Q_0 + \pi_2^*Q_1$ untouched. We do this by deforming the equation $Q(\overline{x})(\lambda_0 F_0(\overline{y}) + \lambda_1 F_1(\overline{y}))$ to

$$g(\overline{x}, \overline{y}) = A_0(\overline{x})F_0(\overline{y}) + A_1(\overline{x})F_1(\overline{y}),$$

where $A_j, j = 1, 2$ are homogeneous polynomial of degree two in X. We use the same letter for the polynomial A_j and for the corresponding quadric curve in $\mathbb{P}^2$. Denote by B_1 the divisor $(g = 0)$. These divisors form an 11-dimensional linear system E_1 on Z with base locus $S_1 \cup S_2$. The general element is thus smooth away from $S_1 \cup S_2$. B_1 is also smooth in a general point $p = (\overline{x}', \overline{y}')$ on S_i. The set of singular values of the projection $\pi_{1|B_1} : B_1 \longrightarrow \mathbb{P}^2$ is a reduced curve Δ_{B_1} of degree six in $\mathbb{P}^2$. For degree reasons the smooth critical points are simple.

I. *Consider the divisor $B_1 + \pi_2^*Q_0$.* The divisor $\pi_2^*Q_0$ itself is smooth and has simple smooth critical points over the curve Q_0^* dual to Q_0. Consider $B_1 \cap \pi_2^*Q_0$. On Z this is given by the equations

$$\begin{aligned} A_1(\overline{x})F_1(\overline{y}) &= 0 \\ F_0(\overline{y}) &= 0. \end{aligned}$$

$B_1 + \pi_2^*Q_0$ contains the lines S_1, S_2.

I.i. *$B_1 \cap \pi_2^*Q_0$ away from $S_1 \cup S_2$.* Away from $S_1 \cup S_2$ $B_1 \cap \pi_2^*Q_0$ is given by

$$\begin{aligned} F_0(\overline{y}) &= 0 \\ A_1(\overline{x}) &= 0. \end{aligned}$$

Over a general point of A_1, $F_0(\overline{y}) = 0$ has two roots. So away from $S_1 \cup S_2$, $B_1 \cap \pi_2^*Q_0$ is generically two-to-one over A_1. Over a neighbourhood in $\mathbb{P}^2$ of a general point of A_1 the divisor $B_1 + \pi_2^* Q_0$ looks like two copies of *node* $\times$ disc.

For a divisor D on Z denote by $\mathrm{Ramif}(D)$ the ramification locus of $\pi_{1|D}$.

I.ii. $B_1 \cap \mathrm{Ramif}(\pi_2^* Q_0)$. The quadric A_1 intersects Q_0^* in four points. Over these four points B_1 intersects the ramification divisor of $\pi_2^* Q_0$. But over these four points B_1 is $F_0(\overline{y}) = 0$, so B_1 too has a smooth critical point there. So over these four points B_1 and $\pi_2^* Q_0$ intersect in a single point that is a smooth critical point for both B_1 and $\pi_2^* Q_0$. Apart from these four points of $\mathrm{Ramif}(B_1) \cap \mathrm{Ramif}(\pi_2^* Q_0)$ over $A_1 \cap Q_0^*$ there are two points on $S_1 \cup S_2$ where B_1 intersects $\mathrm{Ramif}(\pi_2^* Q_0)$. On $S_1 \cup S_2$ the intersection of B_1 with the ramification divisor of $\pi_2^* Q_0$ is given by

$$\begin{aligned} F_0(\overline{y}) &= 0 \\ F_1(\overline{y}) &= 0 \\ F_0^*(\overline{x}) &= 0, \end{aligned}$$

where $F_0^*(\overline{x}) = 0$ is an equation for Q_0^*. Q_0^* is tangent to s_1 and s_2 in the points t_1 and t_2 respectively. So the above set consists in the two points $(t_i, u_i), i = 1, 2$. t_1 and t_2 are the two points over which the line $\pi_1^{-1}(t_i) = H_{i|t_i}$ tangent to $B_1, \pi_2^* Q_0, \pi_2^* Q_1$ is vertical.

I.iii. $\pi_2^* Q_0 \cap \mathrm{Ramif}(B_1)$. Let $\mathrm{pr}_1 : \mathbb{P}^2 \times \mathbb{P}^{2*} \longrightarrow \mathbb{P}^2$ and $\mathrm{pr}_2 : \mathbb{P}^2 \times \mathbb{P}^{2*} \longrightarrow \mathbb{P}^2$ be the two projections. In $\mathbb{P}^2 \times \mathbb{P}^{2*}$ this is given by

$$\begin{aligned} A_1(X) F_1(Y) &= 0 \\ F_0(Y) &= 0 \\ A_0(X) \,\mathrm{grad}_Y F_0(Y) + A_1(X) \,\mathrm{grad}_Y F_1(Y) &= \lambda \cdot X \\ X \cdot Y &= 0 \end{aligned}$$

for some function λ. On Z it is thus given by

$$\begin{aligned} A_1(\overline{x}) F_1(\overline{y}) &= 0 \\ F_0(\overline{y}) &= 0 \\ A_0(\overline{x}) \,\mathrm{grad}_Y F_0(\overline{y}) + A_1(\overline{x}) \,\mathrm{grad}_Y F_1(\overline{y}) &= \lambda \cdot \overline{x}. \end{aligned}$$

Outside of $S_1 \cup S_2$ this is

$$\begin{aligned} A_1(\overline{x}) &= 0 \\ F_0(\overline{y}) &= 0 \\ A_0(\overline{x}) \operatorname{grad}_Y F_0(\overline{y}) &= \lambda \cdot \overline{x}. \end{aligned}$$

The quadrics A_0 and A_1 intersect in four points. Over each of these four points B_1 consists in the whole line Z_b and intersects $\pi_2^* Q_0$ in two points.

If $A_0(\overline{x}) \neq 0$, then

$$\operatorname{grad}_Y F_0(\overline{y}) = \lambda \cdot \overline{x},$$

which means that $(\overline{x}, \overline{y})$ is a critical point of $\pi_{1|\pi_2^* Q_0} : \pi_2^* Q_0 \longrightarrow \mathbb{P}^2$. So these are the four points of $\operatorname{Ramif}(B_1) \cap \operatorname{Ramif}(\pi_2^* Q_0)$, we saw above.

For $B_1 + \pi_2^* Q_1$ one obtains the same description.

II. *Consider now the divisor $B_1 + \pi_2^* Q_0 + \pi_2^* Q_1$.* The discriminant locus of the projection

$$\pi_{1|B_1+\pi_2^* Q_0+\pi_2^* Q_1} : B_1 + \pi_2^* Q_0 + \pi_2^* Q_1 \longrightarrow \mathbb{P}^2$$

consists in the union of $s_1 + s_2$ with $\Delta_{B_1} + Q_0^* + Q_1^*$, which is a reduced curve of degree ten, and $A_0 + A_1$. Over a neighbourhood in $\mathbb{P}^2$ of a general point of $s_1 + s_2$ the divisor $B_1 + \pi_2^* Q_0 + \pi_2^* Q_1$ is isomorphic to the disjoint union of three smooth branches with

$$\textit{simple infinitely close triple point} \times \text{disc}\,.$$

Above a general point of $\Delta_{B_1} + Q_0^* + Q_1^*$ the projection has a single simple smooth crititical point. And over a neighbourhood of a general point of $A_0 + A_1$ the divisor $B_1 + \pi_2^* Q_0 + \pi_2^* Q_1$ is isomorphic to the disjoint union of two smooth branches with two copies of

$$\textit{node} \times \text{disc}\,.$$

After this description we are now ready to prove the proposition. As the general element of E is smooth away from $S_1 \cup S_2$ a generic deformation of $B_1 + \pi_2^* Q_0 + \pi_2^* Q_1$ in E will smooth all singularities of $B_1 + \pi_2^* Q_0 + \pi_2^* Q_1$ outside $S_1 \cup S_2$ and keep the singularities over general points of $s_1 + s_2$. The simple smooth critical points over a general point of $\Delta_{B_1} + Q_0^* + Q_1^*$ deform into simple smooth critical points over a curve Δ_1 of degree 10 in $\mathbb{P}^2$ that is a deformation of $\Delta_{B_1} + Q_0^* + Q_1^*$. The singularities over a general point of A_i on the other hand will smooth into two pairs of simple smooth critical points. This yields a component $\Delta_{2,\mathrm{red}}$ of degree 8 of the discriminant locus such that over a general point of $\Delta_{2,\mathrm{red}}$ lies a pair of simple smooth critical points.

The last two assertions can be seen as follows. Let

$$G(\overline{x}, F_0(\overline{y}), F_1(\overline{y})) = 0$$

be an equation for a general element B of E and let $A_0(\overline{x}), ..., A_3(\overline{x})$ be the coefficients of $G(\overline{x}, W_0, W_1)$. For $\overline{x} \in \mathbb{P}^2$ $G(\overline{x}, W_0, W_1)$ is a homogeneous cubic in $[W_0, W_1] \in \mathbb{P}^1$. Thus

$$G(\overline{x}, W_0, W_1) = 0$$

defines a surface $V(G)$ of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. The discriminant *Discr(G, W)* of G with respect to $W = [W_0, W_1]$ is an irreducible homogeneous polynomial of degree four in the coefficients $A_0, ..., A_3$. For a general choice of $A_0(\overline{x}), ..., A_3(\overline{x})$

$$\mathit{Discr}(\overline{x}) := \mathit{Discr(G, W)}(A_0(\overline{x}), ..., A_3(\overline{x}))$$

is an irreducible homogeneous polynomial of degree eight in $\overline{x}$. For a general point $\overline{x} \in (\mathit{Discr} = 0)$ the cubic $G(\overline{x}, W_0, W_1)$ has two zeros, i.e.

$$G(\overline{x}, W_0, W_1) = (\mu_1 W_0 - \nu_1 W_1)^2 (\mu_2 W_0 - \nu_2 W_1),$$

such that the $[\nu_i, \mu_i] \in \mathbb{P}^1$ are distinct. Consider the map

$$\begin{array}{ccc} Z \setminus Z_{s_1 \cup s_2} & \xrightarrow{\chi} & \mathbb{P}^2 \times \mathbb{P}^1 \\ (\overline{x}, \overline{y}) & \mapsto & (\overline{x}, [F_0(\overline{y}), F_1(\overline{y})]). \end{array}$$

This map is regular and generically two-to-one. For $\overline{x} \notin s_1 \cup s_2$ the morphism

$$\begin{array}{ccc} Z_{\overline{x}} & \xrightarrow{\chi_{\overline{x}}} & \mathbb{P}^1 \\ \overline{y} & \mapsto & [F_0(\overline{y}), F_1(\overline{y})] \end{array}$$

is a double cover branched in two points. Let b be a point in $s_i \setminus \{q\}$, where q is the intersection point of s_1 with s_2, $v = [v_0, v_1]$ homogeneous coordinates on Z_b and $F_{0b}(v), F_{1b}(v)$ the quadrics $F_{0|Z_b}, F_{1|Z_b}$ in the coordinates v. As the line Z_b passes through u_i the quadrics $F_{0b}(v), F_{1b}(v)$ have a common root. So

$$\begin{array}{rcl} F_{0b}(v) & = & (\beta v_0 - \alpha v_1)(\delta v_0 - \gamma v_1) \\ F_{1b}(v) & = & (\beta v_0 - \alpha v_1)(\vartheta v_0 - \varepsilon v_1) \end{array}$$

for pairwise distinct $[\alpha,\beta],[\gamma,\delta],[\varepsilon,\vartheta]\in\mathbb{P}^1$. Consequently the map

$$\begin{array}{ccc} Z_b & \xrightarrow{\chi_b} & \mathbb{P}^1 \\ \overline{y} & \mapsto & [F_0(\overline{y}),F_1(\overline{y})] \end{array}$$

defines an isomorphism. This extends the morphism $\chi : Z\setminus Z_{s_1\cup s_2}\longrightarrow\mathbb{P}^2\times\mathbb{P}^1$ to a surjective morphism

$$\chi : Z\setminus Z_q\longrightarrow(\mathbb{P}^2\setminus\{q\})\times\mathbb{P}^1.$$

This morphism is generically two-to-one. We denote its branch locus by V. The closure $\overline{V}$ is a surface in $\mathbb{P}^2\times\mathbb{P}^1$.

Let $J\subset\mathbb{P}^2\times\mathbb{P}^1$ be defined by

$$\begin{array}{rcl} G & = & 0 \\ Discr & = & 0. \end{array}$$

For general $A_0,...,A_3$ the curve J intersects $\overline{V}$ in finitely many points. Therefore if $\overline{x}$ is a general point on $(Discr=0)$, then

$$G(\overline{x},F_0,F_1)=(\mu_1F_0(\overline{y})-\nu_1F_1(\overline{y}))^2(\mu_2F_0(\overline{y})-\nu_2F_1(\overline{y}))$$

has exactly four zeros, two of them twofold and two of them simple. It follows that $\pi_{1|B} : B\longrightarrow\mathbb{P}^2$ has two simple smooth critical points over a general point $\overline{x}\in(Discr=0)$. The curve $(Discr=0)$ is the curve $\Delta_{2,\mathrm{red}}$.

As for Δ_1 consider again the morphism χ. The surfaces $\overline{V}$ and $V(G)$ in $\mathbb{P}^2\times\mathbb{P}^1$ intersect in a curve $\overline{V}\cap V(G)$. A general point on $\overline{V}\cap V(G)$ is both a simple zero of G and a branch point of χ. Thus B has a smooth critical point in the fibre $Z_{\overline{x}}$.

The surface $\overline{V}$ has bidegree $(4,2)$ in $\mathbb{P}^2\times\mathbb{P}^1$. That it has bidegree $(\cdot,2)$ is clear. That it has bidegree $(4,2)$ can be seen as follows. A point $(\overline{x},p)$ is a branch point of $\chi : Z\setminus Z_{\overline{q}}\longrightarrow(\mathbb{P}^2\setminus\{q\})\times\mathbb{P}^1$ either if $\chi_{\overline{x}} : Z_{\overline{x}}\longrightarrow\mathbb{P}^1$ is an isomorphism, i.e $\overline{x}\in s_1\cup s_2$, or if $\chi_{\overline{x}} : Z_{\overline{x}}\longrightarrow\mathbb{P}^1$ is $2:1$ and branched over $p=[\lambda_0,\lambda_1]$. In the second case $\chi_{\overline{x}}$ is on $Z_{\overline{x}}$ given by

$$[v_0,v_1]\mapsto[F_0(v),F_1(v)].$$

Assume $\lambda_1\neq 0$. Then p is a branch point if and only if $\lambda_1F_{0\overline{x}}+\lambda_0F_{1\overline{x}}=0$ has a twofold root $v\in Z_{\overline{x}}$. If we denote the quadric $\lambda_1F_0+\lambda_0F_1=0$ in $\mathbb{P}^{2*}$ by $Q_{[\lambda_0,\lambda_1]}$, then such a v exists if and only if $\overline{x}$ lies on the dual quadric $Q^*_{[\lambda_0,\lambda_1]}$. From this we see that

$$V \cap \big((\mathbb{P}^2 \setminus \{q\}) \times \{p\}\big) = \big(Q^*_{[\lambda_0,\lambda_1]} \cup ((s_1 \cup s_2) \setminus \{q\})\big) \times \{p\}.$$

This implies that $\overline{V}$ has bidegree $(4,2)$. The surface $V(G)$ on the other hand has bidegree $(2,3)$. For a general line L in $\mathbb{P}^2$, the intersections of the curves $V_{|L}$ and $V(G)_{|L}$ are transverse. As these curves have bidegree $(4,2)$ and $(2,3)$ respectively on $L \times \mathbb{P}^1$, they intersect in 16 points. Three of these points lie above $L \cap s_1$ and three above $L \cap s_2$. The three points lying over $L \cap s_i$ do not correspond to critical points of the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^2$. The fibre of $\pi_{1|B}$ over $L \cap s_i$ contains a threefold point and three simple points of B. The three points $(V \cap V(G))_{|L\cap s_i}$ correspond to the simple points. The remaining ten points of $(V \cap V(G))_{|L}$ however do correspond to smooth critical points and as the A_i are general these ten points lie in different fibres. Consequently $\Delta_1 := \overline{\pi_1((V \cap V(G))_{|\mathbb{P}^2 \setminus (s_1 \cup s_2)})}$ is a curve of degree 10 in $\mathbb{P}^2$ and over a general point of Δ_1 lies a single simple smooth critical point.

□

Remark 3.43. The set D_{red} of critical values of $\pi_{1|B} : B \longrightarrow \mathbb{P}^2$ is

$$D_{\text{red}} = s_1 + s_2 + \Delta_1 + \Delta_{2,\text{red}}.$$

Let $D_{\text{red,sing}}$ be the singular locus of the curve D_{red}. Over a point in $\Delta_1 \setminus D_{\text{red,sing}}$ the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^2$ has a single simple smooth critical point. And over a point in $\Delta_{2,\text{red}} \setminus D_{\text{red,sing}}$ the projection $\pi_{1|B} : B \longrightarrow \mathbb{P}^2$ has two simple smooth critical points. Over a general line L in $\mathbb{P}^2$ we are therefore in the situation of Proposition 3.39.

From a divisor B on Z as in Proposition 3.42 one can analogously to the proof of proposition 3.37 construct a genus two fibration $\mathfrak{f} : Y' \longrightarrow \mathbb{P}^2 \setminus N$ outside a finite set N of points on $s_1 \cup s_2$.

Proposition 3.44. *There exists a genus two fibration $\mathfrak{f} : Y' \longrightarrow \mathbb{P}^2 \setminus N$, where Y' is a smooth threefold and N a finite set in $\mathbb{P}^2$, such that the (reduced) discriminant locus of $\mathfrak{f}$ consists in two lines s_1', s_2', a curve Δ_1' of degree* 10 *and a curve $\Delta_{2,\text{red}}'$ of degree* 8*. Over a general point of $s_1' + s_2'$, Δ_1' and $\Delta_{2,\text{red}}'$ the fibre is a curve with a single separating node, an irreducible curve with a single non-separating node and an irreducible curve with two non-separating nodes respectively.*

Proof: Let $\sigma_a : \widetilde{Z} \longrightarrow Z$ be the blow up of S_1 and S_2. Denote by E_1 and E_2 the two exceptional divisors of σ_a and by $\widetilde{H}_1, \widetilde{H}_2$ the strict transforms of H_1, H_2. If we consider E_i as a $\mathbb{P}^1$-bundle over S_i, then $\widetilde{H}_i \cap E_i$ is a section, which we denote by Σ_i. Let $\sigma_b : \widetilde{\widetilde{Z}} \longrightarrow \widetilde{Z}$ be the blow up of Σ_1 and Σ_2 and denote the strict transform of E_i by Γ_i. Choose affine coordinates (x_1, x_2) on $\mathbb{P}^2 \setminus s_2 \simeq \mathbb{C}^2$ such that $s_1 = \{x_2 = 0\}$. Then $Z_{|\mathbb{C}^2} \simeq \mathbb{C}^2 \times \mathbb{P}^1$ and S_1 is a section of $Z_{|s_1}$. Projection to the x_1-axis exhibts $Z_{|\mathbb{C}^2}$ as a surface fibration over s_1 and $S_1$is a section. Then $\widetilde{Z}_{|\mathbb{C}^2} \longrightarrow s_1$ is this fibration with S_1 blown up and inturn $\widetilde{\widetilde{Z}}_{|\mathbb{C}^2} \longrightarrow s_1$ the blow up of Σ_1 in $\widetilde{Z}_{|\mathbb{C}^2} \longrightarrow s_1$. It follows that the divisor Γ_1 is a family of (-2)-curves in $\widetilde{\widetilde{Z}}_{|\mathbb{C}^2} \longrightarrow s_1$ and analogously for Γ_2. For $i = 1, 2$ let S_i' be the set of points p on S_i such that locally around p B is isomorphic to

$$\textit{simple infinitely close triple point} \times \text{disc}\,.$$

By Proposition 3.42 $S_i \setminus S_i'$ is finite. Let $N := \pi_1((S_1 \setminus S_1') \cup (S_2 \setminus S_2'))$. Now we restrict everything to $\mathbb{P}^{2\prime} := \mathbb{P}^2 \setminus N$ and denote this restriction by a prime, for example $Z' := Z \setminus Z_N$.

As B' has a simple infinitely close triple point along S_i', the strict transform of B' under σ_a has a simple triple point along Σ_i'. Denote by $\widetilde{B}'$ the divisor on $\widetilde{\widetilde{Z}}'$ that is the union of the Γ_i' with the strict transform of B' under $\sigma_a \circ \sigma_b$. This divisor is smooth and analogously to the proof of Proposition 3.37 one can show that it is two-divisible. Therefore we can construct the double cover of $\widetilde{\widetilde{Z}}'$ branched along $\widetilde{B}'$

$$\pi : \widetilde{Y}' \xrightarrow{2:1} \widetilde{\widetilde{Z}}'$$

and the resulting threefold, denoted by $\widetilde{Y}'$, is smooth. As above $\widetilde{Y}'$ can be written as a surface fibration over s_i'. Then the two reduced divisors $\mathcal{E}_i := (\pi^*(\Gamma_i'))_{\mathrm{red}}$ consist then each in a family of (-1)-curves. Therefore they can be simultaneously contracted

$$\mathrm{contr} : \widetilde{Y}' \longrightarrow Y'.$$

The result is a smooth threefold Y' such that

$$\begin{tikzcd} Y' \arrow[r, dashed, "\phi"] & Z' \\ & \widetilde{Z'} \arrow[u, "\sigma_a"'] \\ \widetilde{Y'} \arrow[uu, "\mathrm{contr}"] \arrow[r, "\pi"] & \widetilde{\widetilde{Z'}} \arrow[u, "\sigma_b"'] \end{tikzcd} .$$

Let $I_i := \mathrm{contr}(\mathcal{E}_i)$. Let

$$\mathfrak{f} : Y' \longrightarrow \mathbb{P}^{2\prime}$$

and

$$\widetilde{\mathfrak{f}} : \widetilde{Y'} \longrightarrow \mathbb{P}^{2\prime}$$

be the induced projections. By the construction of Y' ϕ is a rational map

$$\begin{tikzcd} Y' \arrow[rr, dashed, "\phi"] \arrow[dr, "\mathfrak{f}"'] & & Z' \arrow[dl] \\ & \mathbb{P}^{2\prime} & \end{tikzcd}$$

that is generically of degree two over $\mathbb{P}^{2\prime}$ with branch locus B. The indeterminacy locus of ϕ is $I := I_1 \cup I_2$. The induced fibration $\mathfrak{f} : Y' \longrightarrow \mathbb{P}^{2\prime}$ is a genus two fibration. Let $\Delta_1' := \Delta_1 \setminus N$, $\Delta_{2,\mathrm{red}}' := \Delta_{2,\mathrm{red}} \setminus N$ and $s_i' := s_i \setminus N$. Then the (reduced) discriminant locus of $\mathfrak{f}$ is

$$\Delta_{\mathfrak{f}} = \Delta_1' + \Delta_{2,\mathrm{red}}' + s_1' + s_2'.$$

From the properties of B (Proposition 3.42) we see that the fibre over a general point of Δ_1' is a curve with a single non-separating node, whereas the fibre over a general point of $\Delta_{2,\mathrm{red}}'$ is an irreducible curve with two non-separating nodes. The fibre over a general point of $s_1' + s_2'$ is a curve with a single separating node. The union of the separating nodes is I.

$\square$

From this fibration one can construct an abelian fibration as the relative compactified Jacobian of $\mathfrak{f}$. We will only discuss this construction away from the finite set $M := N \cup \Delta_{\mathfrak{f}\mathrm{sing}}$ in $\mathbb{P}^2$, where $\Delta_{\mathfrak{f}\mathrm{sing}}$ denotes the singular locus of $\Delta_{\mathfrak{f}}$. Let $\mathbb{P}^{2\prime\prime} := \mathbb{P}^2 \setminus M$, $Y'' := \mathfrak{f}^{-1}(\mathbb{P}^{2\prime\prime})$, $\Delta_1'' := \Delta_1 \setminus M$, $\Delta_{2,\mathrm{red}}'' := \Delta_{2,\mathrm{red}} \setminus M$, $s_i'' := s_i \setminus M$

and $\Delta'' := \Delta_1'' \cup \Delta_{2,\mathrm{red}}'' \cup s_1'' \cup s_2''$. Consider the fibration $\mathfrak{f}'' := \mathfrak{f}_{|Y''} : Y'' \longrightarrow \mathbb{P}^{2''}$ and denote the relative compactified Jacobian thereof by

$$f : X'' \longrightarrow \mathbb{P}^{2''}.$$

We will see below that the fourfold X'' is smooth away from $f^{-1}(\Delta_{2,\mathrm{red}}'')$. The reason that it is smooth above $s_1'' \cup s_2''$ is that there it is the quotient of a vector bundle by a lattice of maximal rank. That it is smooth above Δ_1'' follows from a result of Markushevich Theorem 4 [42], which says that the total space of the relative compactified Jacobian is smooth if each singular curve has no more that one non-separating node. However we do not know whether the total space is smooth above $\Delta_{2,\mathrm{red}}''$ as there each curve has two nodes.

Ignoring this difficulty for a minute it is clear that one wants to compactify X'' to a fourfold over $\mathbb{P}^2$ and then resolve the singularities to obtain a smooth fourfold X over $\mathbb{P}^2$ which is then a candidate for a Lagrangian fibration. We note that the abelian fibration obtained in this way is a principally polarised fibration with unipotent monodromy and discriminant locus Δ of degree

$$\deg(\Delta_1) + 2\deg(\Delta_{2,\mathrm{red}}) = 26.$$

The monodromy around Δ_1 is then a simple transvection whereas the monodromy around $\Delta_{2,\mathrm{red}}$ is a the product of two commuting simple transvections.

Consider the above mentioned difficulty that, because of the fact that in this construction there is a component of the discriminant locus over which each curve has two nodes, the relative compactified Jacobian might not be smooth above this part of the discriminant locus. We think it is possible to improve the construction to remedy this difficulty. Namely it should be possible to deform B (beyond the linear system E) into a new divisor B_{new} on Z in such a way that B_{new} retains the properties of B except that $\Delta_2 := 2\Delta_{2,\mathrm{red}}$ is deformed to a reduced curve $\Delta_{2,\mathrm{new}}$ of degree 16 such that B_{new} has a single simple smooth critical point over a general point of $\Delta_{2,\mathrm{new}}$. The assumption that such a B_{new} exists is reasonable, for in the case of fibred surfaces we saw in Proposition 3.40 that there are appropriate divisors on $\mathbb{F}_1$ such that all critical points lie in different fibres.

Conjecture 3.45. There exists a divisor B_{new} of bidegree $(2, 6)$ on Z that satisfies the following conditions. B_{new} contains the lines S_1, S_2 and its only singularities occur on S_1 and S_2. Locally around a general point of S_i it is isomorphic to

$$\textit{simple infinitely close triple point} \times \text{disc}\,.$$

The critical points of $\pi_{1|B_{\text{new}}} : B_{\text{new}} \longrightarrow \mathbb{P}^2$ away from $S_1 \cup S_2$ are as follows. There is a curve Δ_{new} of degree 26 in $\mathbb{P}^2$ such that over a general point of Δ_{new} lies a single simple smooth critical point.

Assume that B_{new} is such a divisor and let D be the discriminant locus of $\pi_{1|B_{\text{new}}} : B_{\text{new}} \longrightarrow \mathbb{P}^2$. Then

$$D = \Delta_{\text{new}} + s_1 + s_2,$$

for a reduced curve Δ_{new} of degree 26. As above one can construct from B_{new} a threefold Y' with a rational map

$$\begin{tikzcd} Y' \arrow[rr, dashed, "\phi"] \arrow[dr, "\mathfrak{f}"'] & & Z' \arrow[dl, "\pi_1"] \\ & \mathbb{P}^{2\prime}. & \end{tikzcd}$$

that is generically of degree two and branched along B_{new}. Consider as before the fibration $\mathfrak{f}'' : Y'' \longrightarrow \mathbb{P}^{2''}$. We claim that in this case the relative compactified Jacobian would be smooth.

Lemma 3.46. *The total space X'' of the relative compactified Jacobian constructed from a divisor $B_{\text{new}} \subset Z$ (as in Conjecture 3.45) is smooth.*

Proof: The discriminant locus of the genus two fibration $\mathfrak{f}'' : Y'' \longrightarrow \mathbb{P}^{2''}$ has three components:

$$\Delta_{\mathfrak{f}''} = \Delta''_{\text{new}} + s''_1 + s''_2.$$

The fibres over Δ''_{new} are curves with a single non-separating node, whereas the fibres over $s''_1 \cup s''_2$ have a single separating node.

Over $\mathbb{P}^{2''} \setminus \Delta_{\mathfrak{f}''}$ the relative Jacobian can be defined as the relative Pic^0, i.e. as the quotient of the rank two vector bundle $R^1\mathfrak{f}''_*\mathcal{O}_{Y''|\mathbb{P}^{2''}\setminus\Delta_{\mathfrak{f}''}}$ by the lattice bundle corresponding to the subsheaf $R^1\mathfrak{f}''_*\mathbb{Z}_{|\mathbb{P}^{2''}\setminus\Delta_{\mathfrak{f}''}}$. The genus two fibration $\mathfrak{f}'' : Y'' \longrightarrow \mathbb{P}^{2''}$ is flat and thus the sheaf $R^1\mathfrak{f}''_*\mathcal{O}_{Y''|\mathbb{P}^{2''}}$ locally free.

The monodromy of $\mathfrak{f}''$ around s''_i lies in the Torelli group as the vanishing cycle is separating. Consequently around s''_i the action of the monodromy on the homology and therefore the monodromy of the local system $R^1\mathfrak{f}''_*\mathbb{Z}_{|\mathbb{P}^{2''}\setminus\Delta_{\mathfrak{f}''}}$ is trivial. From this we infer that the sheaf $R^1\mathfrak{f}''_*\mathbb{Z}_{|\mathbb{P}^{2\prime\prime}\setminus\Delta''_{\mathrm{new}}}$ is a local system of lattices of rank four. So above $\mathbb{P}^{2\prime\prime}\setminus\Delta''_{\mathrm{new}}$ the relative Jacobian can be constructed as

$$\frac{R^1\mathfrak{f}''_*\mathcal{O}_{Y''|\mathbb{P}^{2\prime\prime}\setminus\Delta''_{\mathrm{new}}}}{R^1\mathfrak{f}''_*\mathbb{Z}_{|\mathbb{P}^{2\prime\prime}\setminus\Delta''_{\mathrm{new}}}}$$

and is therefore a smooth family of abelian surfaces.

Over Δ''_{new} the fibres of the relative Jacobian of $\mathfrak{f}'' : Y'' \longrightarrow \mathbb{P}^{2\prime\prime}$ are not compact. But the Altman-Kleiman compactification provides a relative compactification, see [1]. The Altman-Kleiman compactification requires that the fibres be irreducible. This assumption holds for the fibration

$$\mathfrak{f}''_{|Y''_{|\mathbb{P}^{2\prime\prime}\setminus(s''_1\cup s''_2)}} : Y''_{|\mathbb{P}^{2\prime\prime}\setminus(s''_1\cup s''_2)} \longrightarrow \mathbb{P}^{2\prime\prime}\setminus(s''_1\cup s''_2). \tag{41}$$

By a result of Markushevich, Theorem 4 in [42] the total space of the Altman-Kleiman compactification is smooth under a condition that Markushevich calls "mild degenerations", see Definition 4 in [42]. This condition is in particular satisfied by a family $\mathfrak{f} : \mathcal{C} \longrightarrow S$ a genus two fibration over a surface S that has a smooth total space $\mathcal{C}$ and is such that the fibres are irreducible with at most one node. The total space of the genus two fibration (41) is by construction the double cover of a $\mathbb{P}^1$-bundle branched along a smooth divisor and therefore itself smooth. Furthermore all singular fibres are irreducible and have a single node. Consequently the Altmann-Kleimann compactification gives a relative compactified Jacobian of $\mathfrak{f}''_{|Y''_{|\mathbb{P}^{2\prime\prime}\setminus(s''_1\cup s''_2)}}$ that has a smooth total space. Over the smooth locus $\mathbb{P}^{2\prime\prime}\setminus\Delta_{\mathfrak{f}''}$ this construction is isomorphic to the relative Pic^0. Therefore we obtain a relative compactified Jacobian of $\mathfrak{f}'' : X'' \longrightarrow \mathbb{P}^{2\prime\prime}$ that has a smooth total space.

□

We do not know whether X'' would admit a holomorphic-symplectic form. If it did then it might be possible to obtain a compact hyperkähler manifold form X''. Namely suppose that it is possible to compactify X'' to a smooth fourfold X in such a way that $X\setminus X''$ has codimension two in X, then the holomorphic-symplectic form on X'' would by Hartogg's theorem extend to a holomorphic-symplectic form on X.

3.9. **Sawons results and a dichotomy.** Using different techniques Sawon studies in [64] Lagrangian fibrations whose fibres are polarised abelian varieties. He makes an assumption on the general singular fibre. The assumption is that the general singular fibre is a generic semi-stable degeneration of an abelian variety, see the definition in [64] p.4. A Lagrangian fibration is then "good singular fibres", in case it satisfies this assumption. For four-folds and principal polarisations the assumption means that the general singular fibre is the compactified Jacobian of an irreducible curve with a single non-separating node. This assumption on the singularities of the general fibre implies that the monodromy is simple.

Sawon assumes further that the polarisation of the fibres is induced by a global divisor on the irreducible holomorphic-symplectic manifold. This assumption implies in particular that X is projective. In the case of principally polarised abelian surfaces it means that there is a family of genus two curves embedded into the Lagrangian fibration as a family of theta divisors. Under these assumptions Sawon proves the following formula for the degree of the discriminant locus (in the principally polarised case)

$$\deg(\Delta) = 24(n!\sqrt{\widehat{A}}[X])^{\frac{1}{n}},$$

where $\sqrt{\widehat{A}}[X]$ is the characteristic number that comes from the square root of the $\widehat{A}$-polynomial, i.e. a topological quantity, Theorem 5 in [64]. Furthermore using Guan's bounds on the Betti numbers of four dimensional hyperkähler manifolds [25], Sawon proves that the degree of the discriminant locus is smaller than 32 for 4-dimensional fibrations that satisfy his assumptions.

Let $f : X \longrightarrow \mathbb{P}^2$ be a Lagrangian fibration with principally polarised fibres, general singular fibre the compactified Jacobian of an irreducible curve with a single node and polarisation induced by a global divisor. If X is deformation equivalent to $S^{[2]}$ of a $K3$-surface, then Sawons formula yields

$$\deg(\Delta) = 30.$$

Let $f : X \longrightarrow \mathbb{P}^2$ be a Lagrangian fibration with principally polarised fibres, general singular fibre the compactified Jacobian of an irreducible curve with a single node and the polarisation induced by a global divisor. In a recent paper Sawon [66] proves, again using his formula together with Guan's bounds that in this case

$$\deg(\Delta) \geq 30, \tag{42}$$

see the proof of Corollary 3 in [66]. Note that in the corollary he assumes that all the curves are irreducible, but the proof of the above inequality does not use this

assumption.
For such fibrations we have the following dichotomy.

Theorem 3.47. *Let $f : X \longrightarrow \mathbb{P}^2$ be a Lagrangian fibration with principally polarised fibres, general singular fibre the compactified Jacobian of an irreducible curve with a single node and the polarisation induced by a global divisor. Then one of the following two cases occurs. Either the general fibre is reducible as a p.p.a.s. or*

$$\deg(\Delta) = 30$$

and no fibre is reducible as a p.p.a.s..

Proof: According to Sawons results $\deg(\Delta) \geq 30$. If the general fibre is not reducible as a p.p.a.s., then Corollary 3.25 implies $\deg(\Delta) \leq 30$. So $\deg(\Delta) = 30$. For a line $l \subset \mathbb{P}^2$ that intersects Δ transversally Theorem 3.23 implies $\deg(\varphi^* D_1) = 0$. So over such a line there are no fibres that are reducible as a p.p.a.s. and consequently there are no such fibres at all.

□

Conjecture 3.48. *In the second case of the dichotomy of Theorem 3.47, if $f : X \longrightarrow \mathbb{P}^2$ admits a section, then X is deformation equivalent to the Hilbert scheme $S^{[2]}$ of a $K3$-surface S.*

We indicate why we think this is true. The idea is to adapt the argument of Markushevich in [42], Theorem 5. In the second case of the dichotomy $\deg(\Delta) = 30$. Then Theorem 3.23 implies that the fibres over $B_0 := \mathbb{P}^2 \setminus \Delta$ are either Jacobians of smooth genus two curves or compactified Jacobians of irreducible curves with a single node. As in the proof of Lemma 3.18 one can show that over $B_1 := \mathbb{P}^2 \setminus \Delta$ exists the corresponding family $\mathfrak{f} : \mathcal{C}_1 \longrightarrow B_1$ of curves. The fact that the singular fibres over $\Delta \setminus \Delta_{\text{sing}}$ are compactified Jacobians suggests that $\mathfrak{f} : \mathcal{C}_1 \longrightarrow B_1$ extends to a family of stable curves $\mathfrak{f} : \mathcal{C}_0 \longrightarrow B_0$, though we do not have a rigorous proof for this. One could then use Markushevichs construction, Theorem 4 in [42], of the relative compactified Jacobian $P_0 \longrightarrow B_0$ of $\mathfrak{f}$. As both have a section $P_0 \longrightarrow B_0$ should be isomorphic to $X_0 \longrightarrow B_0$, the part of f over B_0. Thereby $K_{P_0} \sim \mathcal{O}_{P_0}$ and P_0 would obtain a holomorphic-symplectic structure that makes the fibres Lagrangian. Note that there is a natural isomorphism over B_1.

The hyperelliptic involution on $\mathfrak{f} : \mathcal{C}_0 \longrightarrow B_0$ induces a morphism of degree two onto the bundle $\mathbb{P}(\mathfrak{f}_* K_{\mathcal{C}_0/B_0})$. The proofs of Proposition 5 and Lemma 1 in [42] p. 179-181 carry over to this situation. They show that one can replace the bundle $\mathfrak{f}_* K_{\mathcal{C}_0/B_0}$ by $\mathcal{T}_{B_0}$. As in Remark 3.35 this implies that the curves in $\mathcal{C}_0$ form a subset of codimension 2 in a linear system on a $K3$-surface S. The proof of Lemma 4 in [42] gives a bimeromorphic map $S^{[2]} <-- > P_0$ and thus a bimeromorphic map $S^{[2]} <-- > X$.

3.10. **Construction again.** Inequality (42) seems to rule out the construction we were trying out in Section 3.8. But this inequality presupposes that the polarisation of the fibres is induced by a global divisor. This assumption would for example be satisfied if the genus two fibration $\mathfrak{f} : Y \longrightarrow \mathbb{P}^2$ had a section. Using this section one could embed Y into X. This would then be a divisor inducing a relative theta divisor on the abelian fibration $f : X \longrightarrow \mathbb{P}^2$. But it is not clear whether the assumption holds in our construction. So there is still room for our construction to lead to a compact hyperkähler manifold.

On the other hand if it turns out that Sawon's inequality is valid also for fibrations such that the polarisation is not induced by a global divisor, then our construction is bound to break down at some point. The point where it might break down is the existence of a holomorphic-symplectic form or the compactification and subsequent resolution of X''. In particular it might not be possible to produce a smooth compactification X of X'' under the condition that $X \setminus X''$ has codimension two in X. To examine this question a careful analysis of the singularities of B_{new} or B would be necessary. In this case however the the dichotomy of Theorem 3.47 calls for an explanation.

3.11. **Intersection theory on $\overline{\mathcal{M}}_2$.** In [58] Mumford defines intersection theory on $\overline{\mathcal{M}}_g$. He then calculates the intersection product for natural classes in $A(\overline{\mathcal{M}}_g)$. In case $g = 2$ he obtains a complete description of the intersection product on $A(\overline{\mathcal{M}}_2)$. Among other classes he defines the classes $\lambda_1, \delta_0, \delta_1$, see [58],p. 299. The class $\lambda_1 \in A^1(\overline{\mathcal{M}}_2)$ should be thought of as the first Chern class of the Hodge bundle of the universal family over $\overline{\mathcal{M}}_2$. It has the property that for a family of stable curves $f : \mathcal{C} \longrightarrow \mathbb{P}^1$ the degree $\deg(\varphi^*\lambda_1)$, where $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_2$ is the moduli map, gives the first Chern class of the Hodge bundle of f. The classes δ_i are the classes of the divisors D_i. Mumford proves the following relation between these classes

$$10\lambda_1 = \delta_0 + 2\delta_1.$$

From this relation we can also deduce Corollary 3.21.

Alternative proof of Corollary 3.21: Let $f : X \longrightarrow \mathbb{P}^1$ be a family of abelian surfaces as in the Corollary. According to Lemma 3.18 and the discussion thereafter, there exists a family of stable genus two curves $\mathfrak{f} : \overline{Y} \longrightarrow \mathbb{P}^1$ corresponding to f. Let $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}_2}$ be the moduli map associated to $\mathfrak{f}$. Then

$$10\deg(\varphi^*\lambda_1) = \deg(\varphi^*\delta_0) + 2\deg(\varphi^*\delta_1).$$

And this implies

$$10c_1(\widetilde{\mathcal{F}}^1) = \deg(\varphi^* D_0) + 2\deg(\varphi^* D_1)$$

where $\widetilde{\mathcal{F}}^1$ is the Hodge bundle of $f_{X_l} : X_l \longrightarrow l$.

□

3.12. **Two examples with non-unipotent monodromy.** We know two examples of 4-dimensional Lagrangian fibrations other than Example 3.24, whose fibres are principally polarised. Both of them do not have unipotent monodromy of rank one. Consequently Theorem 3.23 fails to apply.

Example 3.49. Let $f : S \longrightarrow \mathbb{P}^1$ be an elliptic $K3$-surface. This induces a Lagrangian fibration on the Hilbert square $S^{[2]}$ that is also discussed in [43]. The fibration $f \times f : S \times S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is equivariant with respect to the $\mathbb{Z}_2$-action permuting the factors. The quotient by this action is a fibration $f^{(2)} : S^{(2)} \longrightarrow (\mathbb{P}^1)^{(2)} = \mathbb{P}^2$ whose general fibre is the product of two elliptic curves. This induces a Lagrangian fibration

$$f^{[2]} : S^{[2]} \longrightarrow \mathbb{P}^2.$$

For simplicity we assume the elliptic fibration on the $K3$-surface S to be Lefschetz, i.e. the singular fibres of f are 24 nodal curves. Let $\Delta_1 = \{p_1, ..., p_{24}\}$ the discriminant locus of f. The discriminant locus of $f \times f$ is then

$$\Delta_{f\times f} = \Delta_1 \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \Delta_1.$$

The map

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\pi} & \mathbb{P}^2 \\ ([x_1, y_1], [x_2, y_2]) & \mapsto & [x_1x_2, x_1y_2 + x_2y_1, y_1y_2] \end{array}$$

is two-to-one and branched along the diagonal D in $\mathbb{P}^1 \times \mathbb{P}^1$. Under π the diagonal D maps to the smooth quadric $Q = (v^2 - 4uw = 0)$ in $\mathbb{P}^2$ and a vertical line $[p, q] \times \mathbb{P}^1$ maps to the line $(q^2u - pqv + p^2w = 0)$. As the corresponding horizontal line $\mathbb{P}^1 \times [p, q]$ maps to the same line, $\Delta_{f\times f}$ maps to a configuration of 24 lines $L_1, ..., L_{24}$ in $\mathbb{P}^2$. These 24 lines are all tangent to the quadric Q and the discriminant locus of $f^{[2]}$ is precisely the union of Q with these 24 tangents. Over a point on $L_i \setminus Q$ the fibre of $f^{[2]}$ is the product of an elliptic curve and a nodal genus one curve. The fibre over a point of Q is the transform of the fibre of $f^{(2)}$ over the same point. For points in $Q \setminus \bigcup_i L_i$ the fibre of $f^{(2)}$ is the symmetric product of the corresponding fibre E_t of f. For $t \in \mathbb{P}^1 \setminus \Delta_1$ the symmetric product $\mathcal{E}_1 := E_t^{(2)}$ is *via*

$$E_t^{(2)} \xrightarrow{+} E_t$$

a ruled surface over E_t. A fibre of $f^{(2)}$ is two-fold as the following local calculation shows. Let (x, y) be affine coordinates on S such that $f : S \longrightarrow \mathbb{P}^1$ is given by $(x, y) \mapsto x$. Then $f \times f : S \times S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is locally given by $(x, y, z, w) \mapsto (x, z)$ and the $\mathbb{Z}_2$-action by $(x, y, z, w) \mapsto (z, w, x, y)$. The invariant functions are then generated by

$$u = xz, s = yw, v = x + z, t = y + w \text{ and } r = xy + zw.$$

Consequently $S^{(2)}$ is locally given by the equation

$$sv^2 + ut^2 - vtr - 4us + r^2 = 0$$

and $f^{(2)}$ by $(u, v, s, t, r) \mapsto (u, v)$. As $Q = (v^2 - 4u = 0)$, $f^{(2)*}Q$ is given by $(v_0t - 2r)^2 = 0$. The fibre of $f^{[2]}$ is thus $2\mathcal{E}_1 + \mathcal{E}_2$, where $\mathcal{E}_2$ is a $\mathbb{P}^1$-bundle over the diagonal in $E_t^{(2)}$.

The monodromy transformation associated to a small circle around a general point of L_i is the product of the monodromies of f around the corresponding two points and thus a simple transvection. The monodromy around a general point of

Q on the other hand is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Therefore the monodromy of $f^{[2]} : S^{[2]} \longrightarrow \mathbb{P}^2$ is not unipotent.

Example 3.50. The second example is a fibration on the generalised Kummer variety K_2 and arises in a similar fashion as the fibration in Example 3.49. Let

$$E \longrightarrow T \xrightarrow{f} A$$

be an elliptic torus. This gives a fibration $f^{(3)} : T^{(3)} \longrightarrow A^{(3)}$ that in turn induces a fibration on K_2.

$$\begin{tikzcd}
K_2 \arrow[r, hook] \arrow[d] & T^{[3]} \arrow[d] & \\
K_2' \arrow[r, hook] \arrow[d] & T^{(3)} \arrow[r] \arrow[d, "f^{(3)}"] & T \arrow[d] \\
B \arrow[r, hook] & A^{(3)} \arrow[r] & A
\end{tikzcd}$$

The base of the induced fibration on K_2 is $B := \{\{u, v, w\} \in A^{(3)} | u + v + w = 0\}$. Embed A as a cubic in $\mathbb{P}^2$. Then as three points on a cubic that sum up to zero lie on line in $\mathbb{P}^2$, the base B is indeed a $\mathbb{P}^2$,

$$f_2 : K_2 \longrightarrow \mathbb{P}^2.$$

The general fibre of f_2 is isomorphic to E^2. The discriminant locus Δ of this fibration is the diagonal in B. This corresponds to the set of lines in $\mathbb{P}^2$ that are tangent to A. Thus Δ is the curve that is the plane dual of $A \subset \mathbb{P}^2$. This is a sextic with nine cusps. Reasoning analogously to Example 3.49 shows that the fibre of $f^{(3)}_{|K_2'}$ over a regular point of Δ is isomorphic to the symmetric product $E^{(2)}$ and has multiplicity two. Consequently the singular fibre of f_2 over a regular point of Δ is isomorphic to $2E^{(2)} + F$, where F is a ruled surface over the diagonal in $E^{(2)}$. Furthermore the monodromy around a regular point of Δ is the same as above. So also in this example the monodromy is not unipotent of rank one. Note that here the moduli map is constant. Note also that by the construction in Example

3.49 one can also obtain a fibration with this property on $S^{[2]}$ provided that one starts with a Kummer-$K3$-surface S that is constructed from an elliptic torus.

Remark 3.51 (Affine structures in the two examples)**.** In both examples the affine structure on $\mathbb{P}^2 \setminus \Delta$ arises as a quotient of an affine manifold. In the first case the original affine manifold is simply the product $\mathbb{P}^1 \setminus \Delta_1 \times \mathbb{P}^1 \setminus \Delta_1$ of the affine structure induced by the elliptic fibration on the $K3$-surface. With respect to this the permutation of the factors is an affine transformation. The fix point set of the corresponding $\mathbb{Z}_2$-action is the diagonal D. On

$$(\mathbb{P}^1 \setminus \Delta_1 \times \mathbb{P}^1 \setminus \Delta_1) \setminus D$$

$\mathbb{Z}_2$ acts properly discontiously by affine transformations. Thus the quotient is an affine manifold.

In the second example Δ is a sextic with nine cusps. The double cover of $\mathbb{P}^2$ branched along Δ is thus a singular $K3$-surface with nine A_2-singularities, where by a singular $K3$-surface we mean a surface with only rational double points such that the minimal resolution is a $K3$-surface. By a theorem of Barth [4] each singular $K3$-surface admits a cyclic triple cover branched in the nine cusps. The triple cover is then a complex torus. In our case the torus is $E \times E$ and the affine structure on $\mathbb{P}^2 \setminus \Delta$ is the quotient of the standard affine structure on $E \times E$ by an S_3 of affine transformations.

3.13. **Discussion.** As we pointed out there is still room for principally polarised Lagrangian fibrations $f : X \longrightarrow \mathbb{P}^2$ with unipotent monodromy of rank one and $\deg(\Delta) < 30$. And it remains an interesting question whether such fibrations exist. Especially as it is not clear whether for such an example X would be deformation equivalent to one of the two standard examples. The only example of a principally polarised Lagrangian fibration on K_2 is Example 3.50, which is a very special fibration. And there are reasons that for fibrations on K_n other types of polarisations are natural, for this see section 5 in [65]. For projective Lagrangian fibrations with X deformation equivalent to K_n a result of Mukai, [65] Proposition 5.3, says that the induced polarisation of the fibres cannot be principal. On the other hand Debarre [17] exhibits a fibration on K_2 with a fibrewise polarisation of type $(1, 3)$.

Theorem 3.23 and Theorem 3.47 are steps towards a classification of 4-dimensional Lagrangian fibrations. We expect that formulas for the degree of the discriminant locus similar to that of Theorem 3.23 and bounds on the degree as in Corollary 3.25 can also be obtained for other types of polarisations.

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