Thick subcategories for quiver representations

Nikolay Dimitrov Dichev

A thesis presented for the degree of
Doktor der Naturwissenschaften (Dr. rer. nat.)

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Abstract

The central objects of investigation in this thesis are the thick subcategories as well as the exact abelian extension closed subcategories of the category of quiver representations. A full additive subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called thick, provided that $\mathcal{C}$ is closed under taking direct summands, kernels of epimorphisms, cokernels of monomorphisms and extensions. The category $\mathcal{C}$ is called exact abelian if it is abelian, the embedding functor preserves exact sequences, hence closed under arbitrary kernels and cokernels.

First we consider the category of locally nilpotent representations over the path algebra of the cyclic quiver. We show that any thick subcategory is exact abelian. Then we give a combinatorial description of thick subcategories via non-crossing arcs on the circle and using generating functions, we calculate their number. Furthermore, we establish a bijection between thick subcategories with a projective generator, thick subcategories without a projective generator, support-tilting and cotilting modules. Then we study exact abelian extension closed subcategories for Nakayama algebras, and we find a recursive formula for their number.

For a finite and acyclic quiver, we consider the category of its quiver representations. We show that any thick subcategory generated by preprojective or preinjective representations is exact abelian. Then we specialise to Euclidian quiver case and we verify that any thick subcategory is exact abelian. Furthermore, we extend a result of Ingalls and Thomas and we give a complete combinatorial classification of thick subcategories in that case.

For a hereditary algebra A, we consider the tilted algebra $B = \mathrm{End}_A(T_A)$, where T_A is a tilting module. We establish a bijection between the exact abelian extension and torsion closed subcategories of $\operatorname{mod} A$ and the exact abelian extension closed subcategories of $\operatorname{mod} B$.

Dicke Unterkategorien für Köcherdarstellungen

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit den dicken, sowie mit den exakten abelschen und erweiterungsabgeschlossenen Unterkategorien der Kategorie der Darstellungen eines Köchers. Eine volle additive Unterkategorie $\mathcal{C}$ einer abelschen Kategorie $\mathcal{A}$ heißt dick, falls $\mathcal{C}$ abgeschlossen ist unter Bildung von direkten Summanden, Kernen von Epimorphismen, Kokernen von Monomorphismen und Erweiterungen. Die Kategorie $\mathcal{C}$ heißt exakt abelsch, falls sie abelsch ist und der Einbettungsfunktor exakte Folgen erhält, insbesondere ist $\mathcal{C}$ dann abgeschlossen bezüglich Bildung von beliebigen Kernen und Kokernen.

Zunächst untersuchen wir die Kategorie der lokal nilpotenten Darstellungen über der Wegealgebra eines zyklischen Köchers. Wir zeigen, dass eine dicke Unterkategorie exakt abelsch ist. Hiernach beschreiben wir kombinatorisch die dicken Unterkategorien durch nicht kreuzende Bögen auf einem Kreis und mit Hilfe der erzeugenden Funktionen berechnen wir ihre Anzahl. Weiterhin zeigen wir Bijektionen zwischen den dicken Unterkategorien mit projektivem Generator, den dicken Unterkategorien ohne projektiven Generator, Trägerkipp- und Kokippmoduln. Dann untersuchen wir die exakten abelschen und erweiterungsabgeschlossenen Unterkategorien für Nakayama Algebren und finden eine rekursive Formel für ihre Anzahl.

Danach wenden wir uns der Kategorie der Darstellungen endlicher azyklischer Köcher zu. Wir zeigen, dass dicke Unterkategorien, die von präprojektiven oder preinjektiven Darstellungen erzeugt werden, exakt abelsch sind. Wir untersuchen euklidische Köcher im Speziellen und zeigen, dass dicke Unterkategorie exakt abelsch sind. Dann ergänzen wir ein Ergebnis von Ingalls und Thomas zu einer vollständige kombinatorische Klassifikation der dicken Unterkategorien für diesen Fall.

Für eine erbliche Algebra A betrachten wir die gekippte Algebra $B = \mathrm{End}_A(T_A)$, wobei T_A Kippmodul ist. Wir zeigen eine Bijektion zwischen den exakten abelschen erweiterungs- und torsionsabgeschlossenen Unterkategorien von $\operatorname{mod} A$ und den exakten abelschen erweiterungsabgeschlossenen Unterkategorien von $\operatorname{mod} B$.

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Contents

Chapter 1

Introduction

In investigations of the structure and properties of algebras (resp. their modules), it is often essential to have a concrete realisation of a given algebra (resp. their modules). In general, the aim of the representation theory of algebras is to develop tools for such realisations. Due to work of Gabriel [Gab1], each finite dimensional algebra over an algebraically closed field k corresponds to a graphical structure, called a *quiver*, and conversely, each quiver, more precisely its associated *path algebra*, corresponds to an associative k-algebra, which has an identity and it is finite dimensional under some conditions. In fact, using the quiver associated to an algebra A, it is possible to visualise a finitely generated A-module as a *quiver representation*, a family of finite dimensional k-vector spaces, connected by linear maps.

In the thesis, we deal mostly with finite dimensional *hereditary algebras*. An algebra is hereditary, if any submodule of a projective module (=a module with basis vectors) is projective. In fact, any such algebra is realised by the path algebra of a finite and acyclic quiver. The working environment for us is the *module category* of a finite dimensional (hereditary) algebra, that is the category of finite dimensional vector spaces with scalars from the algebra.

Quiver-theoretical techniques provide a convenient way to visualise finite dimensional algebras. However, actually to compute the indecomposable modules and the homomorphisms between them, we need other tools. For a finite dimensional algebra A, there is a special quiver, called the *Auslander-Reiten quiver* of $\operatorname{mod} A$, that combinatorially encodes the building blocks of $\operatorname{mod} A$, namely the indecomposable modules and the irreducible morphisms. It can be considered as a first approximation of the module category of a finite dimensional algebra.

The central objects of our study are thick and exact abelian extension closed subcategories of a module category of an algebra (or equivalently the category of its quiver representations).

A full additive subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called *thick*, provided that $\mathcal{C}$ is closed under taking direct summands, kernels of epimorphisms, cokernels of monomorphisms and extensions. $\mathcal{C}$ is called *exact abelian* if it is abelian, the embedding functor preserves exact sequences, hence closed under arbitrary kernels and cokernels. From the definition it follows that an exact abelian subcategory is thick if and only if it is closed under taking extensions, and a thick subcategory is exact abelian if and only if it is closed under taking arbitrary kernels. The latter is true since if $\mathcal{C}$ is thick, and X, Y are objects in $\mathcal{C}$,

$$\begin{tikzcd}
 & X \arrow[rr, "f"] \arrow[dr, two heads] & & Y \arrow[dr, two heads] & \\
\operatorname{Ker} f \arrow[ur, hook] & & \operatorname{Im} f \arrow[ur, hook] & & \operatorname{Coker} f
\end{tikzcd}$$

then $\operatorname{Ker} f \in \mathcal{C} \Leftrightarrow \operatorname{Im} f \in \mathcal{C} \Leftrightarrow \operatorname{Coker} f \in \mathcal{C}$.

The study of exact abelian extension closed subcategories was highlighted by recent work of Colin Ingalls and Hugh Thomas. They establish a large class of bijections involving them, which give a relation to important objects of representation theory of finite dimensional algebras, as well as a relation to recently developing cluster algebras and cluster categories.

Theorem 1.0.1 *[IT] Let Q be a finite acyclic quiver. There are bijections between the following objects:*

- *clusters in the acyclic cluster algebra with initial seed Q;*
- *isomorphism classes of basic cluster-tilting objects in the cluster category;*
- *isomorphism classes of basic support-tilting objects in* $\operatorname{mod} kQ$*;*
- *torsion classes in* $\operatorname{mod} kQ$ *with a projective generator;*
- *exact abelian extension closed subcategories in* $\operatorname{mod} kQ$ *with a projective generator.*

Further, a connection with derived categories was found by Kristian Brüning in his thesis.

Theorem 1.0.2 *[Br1] There is a bijection between thick subcategories in* $D^b(\operatorname{mod} kQ)$ *and exact abelian extension closed subcategories in* $\operatorname{mod} kQ$*.*

In this thesis, the study of exact abelian extension closed subcategories of a hereditary abelian category is continued. I shall give a brief account of my work by outlining the obtained results.

The core work of the thesis is contained in chapters 2, 3 and 4. Each chapter begins with a short introduction. In order to be self-contained, all the facts needed (with appropriate references) are also exposed within the chapter.

In *chapter 2*, we consider the path algebra $k\tilde{\Delta}_n$ of the cyclic quiver,

$$\begin{tikzcd} \tilde{\Delta}_n : 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n. \arrow[llll, bend left] \end{tikzcd}$$

and two (full, additive) subcategories of category of their representations, namely $\tilde{\mathcal{T}}_n$ the category of locally nilpotent representations and $\mathcal{T}_n$ the category of nilpotent representations. We comment that the category $\mathcal{T}_n$ plays an important rôle in the representation theory of algebras of infinite representation type, since it describes (connected) components of the Auslander-Reiten quiver of their module categories.

In proposition 2.2.10, we observe that every thick subcategory in $\mathcal{T}_n$ is exact abelian. After that, in proposition 2.2.13 we give a combinatorial classification of thick subcategories via establishing a bijection with the non-crossing arcs on the circle. Further, in proposition 2.4.2 using generating functions we calculate their number.

The main result in the chapter is a bijection involving thick subcategories.

Theorem 1.0.3 *There is a bijective correspondence between:*

- *isomorphism classes of support-tilting objects in $\mathcal{T}_n$;*
- *thick subcategories in $\mathcal{T}_n$ with a projective generator;*
- *thick subcategories in $\mathcal{T}_n$ without a projective generator;*
- *isomorphism classes of cotilting objects in $\tilde{\mathcal{T}}_n$.*

At the end, we classify exact abelian extension closed subcategories for a class of algebras, called Nakayama algebras, which are quotients of the path algebra of the cyclic quiver. In proposition 2.6.13, we give a recursive formula for their number. We comment that the found formula is a generalisation of the recursive formula for the Catalan numbers.

The results in *Chapter 3* are joint work with Yu Ye. For a finite and acyclic quiver Q, we consider its path algebra kQ. We step on a result of Crawley-Boevey [CB1, Lemma 5], which says that any thick subcategory of $\operatorname{mod} kQ$ generated by an exceptional sequence (a special sequence of indecomposable kQ-modules) is exact abelian. In proposition 3.1.10, we construct for a thick subcategory $\mathcal{C} \subseteq \operatorname{mod} kQ$ generated by preprojective modules, an exceptional sequence that generates $\mathcal{C}$. After that we specialise to the module category of kQ, where Q is an Euclidian quiver. We introduce reduction techniques, some of which work in a more general context (see proposition 3.2.12), which enable us to prove that any thick subcategory in mod kQ is exact abelian (theorem 3.2.14). Further, by a result of Colin Ingalls and Hugh Thomas [IT, Theorem 1.1], there is a bijection between non-crossing partitions associated to Q and exact abelian extension closed subcategories with a projective generator in mod kQ. As one observes, there are exact abelian extension closed subcategories without a projective generator (for instance the tubes in the regular component of the Auslander-Reiten quiver of mod kQ). So we use results from the second chapter and combining with the above cited theorem, we give a complete classification.

Theorem 1.0.4 *Let k be an algebraically closed field, Q an Euclidian quiver and $\mathcal{C}$ a connected exact abelian extension closed subcategory of* mod kQ*.*

(i) [IT] If $\mathcal{C}$ has a projective generator, then $\mathcal{C}$ corresponds to a non-crossing partition of type Q.

(ii) If $\mathcal{C}$ has no projective generator, then $\mathcal{C}$ corresponds to a configuration of non-crossing arcs covering the circle.

At the end of the chapter, we present a very elegant proof, due to Dieter Vossieck, that every thick subcategory of a hereditary abelian category is exact abelian.

In *chapter 4* we deal with *tilted algebras*, an important class of algebras which have been extensively studied in [Bo] and [HaR]. For a finite dimensional hereditary algebra A, there is the concept of a tilting module T_A, which can be thought of as being close to the Morita progenerator. If we consider the k-algebra $B = \operatorname{End}_A(T_A)$, then the categories mod A and mod B are reasonably close to each other. The algebra $B = \operatorname{End}_A(T_A)$ is called tilted algebra. The benefit of tilted algebras is that when the representation theory of an algebra A is difficult to study directly, it may be convenient to replace A with the simpler algebra $B = \operatorname{End}_A(T_A)$, and then to reduce the problem on mod A to a problem on mod B.

The main result in the chapter is a classification of exact abelian extension closed categories for tilted algebras.

Theorem 1.0.5 *Let A be a finite dimensional hereditary k-algebra, T_A a basic tilting module and $B = \operatorname{End}_A(T_A)$. Then there is a bijection between the exact abelian extension and torsion closed subcategories of* mod A *and the exact abelian extension closed subcategories of* mod B*.*

The thesis end with an *Appendix*, where some basic facts, relevant to all chapters, are collected.

Chapter 2

Thick subcategories for cyclic quivers

This chapter is dedicated to study thick subcategories for the category of locally nilpotent cyclic quiver representations. We establish a bijection involving thick subcategories, cotilting and support-tilting objects of that category. Further, we present a combinatorial classification of thick subcategories as well as we calculate their number. At the end, we investigate the exact abelian extension closed categories for algebras which are quotients of the path algebra of the cyclic quiver.

2.1 Cyclic quivers

In the whole chapter k is an algebraically closed field. We begin with very general framework and consider categories which are k-linear, small abelian, Hom-finite, hereditary and satisfy Serre duality. Following [Ln], we recall shortly all these concepts and then specialise to particular examples of such categories, which are target of our investigations.

Let $\mathcal{T}$ be an abelian k-linear category. Recall that k-linearity of $\mathcal{T}$ means that the morphism groups are k-vector spaces, and that composition

$$\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z), (g, f) \mapsto gf,$$

is k-bilinear for all objects X, Y and Z from $\mathcal{T}$.

We recall the notion of an **abelian category**. By definition, a sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ is called **short exact** if for each object X of $\mathcal{T}$ the induced sequence $0 \to \mathrm{Hom}(X, A) \xrightarrow{\mathrm{Hom}(X,u)} \mathrm{Hom}(X, B) \xrightarrow{\mathrm{Hom}(X,v)} \mathrm{Hom}(X, C)$ is exact and dually for each object Y of $\mathcal{T}$ the sequence $0 \to \mathrm{Hom}(C, Y) \xrightarrow{\mathrm{Hom}(v,Y)} \mathrm{Hom}(B, Y) \xrightarrow{\mathrm{Hom}(u,Y)} \mathrm{Hom}(A, Y)$ is exact. For $\mathcal{T}$ to be abelian, one requires two things:

(1) For every morphism $A \xrightarrow{f} B$ there exist two short exact sequences $0 \to K \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$ and $0 \to C \xrightarrow{\gamma} B \xrightarrow{\delta} D \to 0$ such that f is obtained from the commutative diagram below:

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & & & C \arrow[ur] \arrow[dr, "\gamma"] & & & \\
 & & A \arrow[ur, "\beta"] \arrow[rr, "f"] & & B \arrow[dr, "\delta"] & & \\
 & K \arrow[ur, "\alpha"] & & & & D \arrow[dr] & \\
0 \arrow[ur] & & & & & & 0.
\end{tikzcd}$$

(2) $\mathcal{T}$ has **finite direct sums**, which implies the uniqueness of the additive structure.

We impose on $\mathcal{T}$ some finiteness assumptions: $\mathcal{T}$ is a **small category**, that is, the objects of $\mathcal{T}$ form a set, and $\mathcal{T}$ is **Hom-finite**, that is, all morphism spaces $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ are finite dimensional over k.

The properties of $\mathcal{T}$ so far imply that $\mathcal{T}$ is a **Krull-Schmidt category**.

Proposition 2.1.1 *Each abelian* Hom-*finite* k-*category is a Krull-Schmidt category, that is,*

(i) each indecomposable object from $\mathcal{T}$ *has a local endomorphism ring, and*

(ii) each object from $\mathcal{T}$ *is a finite direct sum of indecomposable objects.*

We assume that the category $\mathcal{T}$ is **hereditary**, that is, the extensions $\mathrm{Ext}^n_{\mathcal{T}}(X, Y)$ vanish in degrees $n \geq 2$ for all objects X, Y from $\mathcal{T}$, see also A.2. Later we shall use that exact abelian subcategory of a hereditary category is again hereditary.

We continue with strengthening the heredity condition, namely, we assume the existence of an **equivalence** $\tau : \mathcal{T} \to \mathcal{T}$ and of natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{T}}(X, Y) \xrightarrow{\sim} D \,\mathrm{Hom}_{\mathcal{T}}(Y, \tau X)$$

for all objects X, Y from $\mathcal{T}$. The consequences of a **Serre duality** are of major importance:

Proposition 2.1.2 *Assume that* $\mathcal{T}$ *is an abelian* k-*category which is* Hom-*finite and satisfies Serre duality. Then the following holds:*

(i) $\mathcal{T}$ *is an* ***Ext-finite*** *hereditary category without non-zero projectives or injectives.*

(ii) $\mathcal{T}$ *has* ***almost split sequences*** *with* τ *acting as the* ***Auslander-Reiten translation****. That is, for each indecomposable object* X *there is an almost-split sequence* $0 \to \tau X \to E \to X \to 0$.

We assume that $\mathcal{T}$ is a **length category**, that is, each object of $\mathcal{T}$ has finite length.

An object U of an abelian category is called **uniserial** if the subobjects of U are linearly ordered by inclusion and form a finite chain

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{\ell-1} \subseteq U_\ell = U.$$

If all indecomposables in an abelian length category $\mathcal{U}$ are uniserial, we call $\mathcal{U}$ **uniserial category**.

The following theorem, due to Gabriel, unifies all notions used up-to-now.

Theorem 2.1.3 *[Gab2, Proposition 8.3] Let* $\mathcal{T}$ *be a* Hom*-finite hereditary length category with Serre duality. Then* $\mathcal{T}$ *is uniserial. Moreover, for the indecomposable objects* $\operatorname{ind}\mathcal{T}$ *of* $\mathcal{T}$*, we have* $\text{ind-}\mathcal{T} = \bigsqcup_{\lambda \in I} \mathcal{T}_\lambda$*, where the Auslander-Reiten quiver of* $\mathcal{T}_\lambda$ *is of the form* $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$*, where* $n \in \mathbb{N}_0$.

Therefore the Auslander-Reiten quiver of $\mathcal{T}$ decomposes into stable tubes, where for convenience $\mathbb{Z}\mathbb{A}_\infty$ is also viewed as a tube of an infinite period.

Now, we introduce the main example of our investigation in this chapter, namely the categories that satisfy all the conditions of the Gabriel's theorem. Before that, we refer the reader to A.1 for recalling basic facts about quivers and their representations. We consider the path algebra $k\tilde{\Delta}_n$ of the **cyclic quiver**:

$$\begin{tikzcd} \tilde{\Delta}_n : 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n. \arrow[llll, bend left] \end{tikzcd}$$

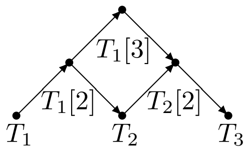
Let $R = R_{\tilde{\Delta}_n}$ be the two-sided ideal generated by all arrows of $\tilde{\Delta}_n$. A $k\tilde{\Delta}_n$-module M is R**-nilpotent** (nilpotent for short) if for each $m \in M$ there exist $\ell \geq 0$ such that $R^\ell.m = 0$. If $\ell = \ell(m)$ depends on m, we say that M is **locally** R**-nilpotent** (locally nilpotent for short). We denote by $\mathrm{nrep}(\mathrm{k}\tilde{\Delta}_\mathrm{n})$ the category of nilpotent and by $\mathrm{NRep}(\mathrm{k}\tilde{\Delta}_\mathrm{n})$ the category of locally nilpotent modules over $k\tilde{\Delta}_n$. If we consider the category of finite dimensional locally nilpotent modules over $k\tilde{\Delta}_n$, we notice that it is the same as $\mathrm{nrep}(\mathrm{k}\tilde{\Delta}_\mathrm{n})$. The argument is the following: Trivially, every nilpotent module is locally nilpotent. Now, if $\ell(M) < \infty$, then for any $\ell > \ell(M)$, R^ℓ annihilates M.

Remark 2.1.4 If we consider the subquiver

$$\begin{tikzcd} \Delta_n : 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n, \end{tikzcd}$$

of $\tilde{\Delta}_n$, then we have a fully-faithful embedding $\operatorname{mod} k\Delta_n \hookrightarrow \operatorname{mod} k\tilde{\Delta}_n$. The quiver Δ_n is the directed $\mathbb{A}_n$ Dynkin quiver. The construction of the Auslander-Reiten quiver of $\operatorname{mod} k\Delta_n$ is well-known, see A.3 for more details.

Example 2.1.5 The Auslander-Reiten quiver of $\operatorname{mod} k\Delta_3$.



We comment that the category of nilpotent modules plays an important rôle in the representation theory of algebras of infinite representation type, since it describes (connected) components of the Auslander-Reiten quiver of their module categories.

For convenience, from now onwards, we denote with $\mathcal{T}_n$ the category of nilpotent modules and with $\tilde{\mathcal{T}}_n$ the category of locally-nilpotent modules over $k\tilde{\Delta}_n$. The following proposition collects all the properties of $\mathcal{T}_n$ so far.

Proposition 2.1.6 *$\mathcal{T}_n$ is* Hom*-finite hereditary length uniserial category with Serre duality.*

The number of isoclasses of simple objects of an abelian category $\mathcal{A}$ is called the **rank** of $\mathcal{A}$ and we denote it by $\operatorname{rk}(\mathcal{A})$. In $\mathcal{T}_n$ we have n simple modules, and we denote them with $T_1, T_2, \ldots, T_n$. Since $\mathcal{T}_n$ is an uniserial category, any indecomposable object is uniquely determined by its socle and length. We set $T_i[\ell]$ to be the indecomposable module with socle T_i and length ℓ. Recall that the simple composition factors of a module X is called the **support** of X and it is denoted by $\operatorname{supp}(X)$.

The construction of the Auslander-Reiten quiver of $\mathcal{T}_n$ is well-known, see [R2, Chapter 4.6]. As mentioned in the Gabriel's theorem, it is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$.

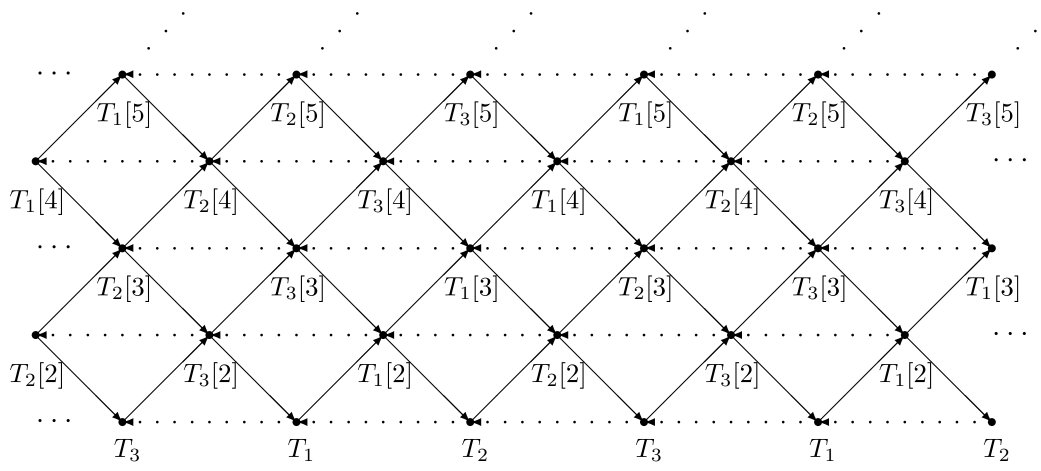


Figure 2.1: AR-quiver of $\mathcal{T}_3$

We consider another category related to $k\tilde{\Delta}_n$, namely the category of **locally finite** modules over the completion algebra $k[[\tilde{\Delta}_n]]$ of $k[\tilde{\Delta}_n] = k\tilde{\Delta}_n$. First, recall that $k[[\tilde{\Delta}_n]] = \underset{\leftarrow}{\lim}\, k[\tilde{\Delta}_n]/R^i$, where R is the same as before. A module is locally finite if it is a filtered colimit of finite length modules. For more details, we refer to the paper of [BKr, Section 2]. Now, we point out the following result.

Theorem 2.1.7 *[CY, Main Theorem] The category of locally finite modules over* $k[[\tilde{\Delta}_n]]$ *is equivalent to* $\mathrm{NRep}(\mathrm{k}\tilde{\Delta}_{\mathrm{n}})$.

In [BKr], the classification of indecomposable objects in the category of locally finite modules over $k[[\tilde{\Delta}_n]]$ and hence in $\mathrm{NRep}(\mathrm{k}\tilde{\Delta}_{\mathrm{n}})$ is made and we shall use it later. We refer the reader to the paper [RV], where complete classification of categories sharing the same properties as $\mathcal{T}_n$ is made.

2.2 Orthogonal sequences and thick subcategories

From now onwards, $\mathcal{T}_n$ will be a tube of rank n. We begin with recalling the following lemma.

Lemma 2.2.1 *[Happel-Ringel] Let* $\mathcal{H}$ *be a hereditary abelian category. Assume that* $X, Y \in \mathcal{H}$ *are indecomposable objects and* $\mathrm{Ext}^1_{\mathcal{H}}(Y, X) = 0$. *Then any non-zero morphism* $f\colon X \to Y$ *is either monomorphism or epimorphism.*

The proof can be found in [AS, Chapter VIII.2, Lemma 2.5]. Now, we make the following observation.

Lemma 2.2.2 *Let* $\zeta : 0 \to X \to Y \to Z \to 0$ *be a non-split short exact sequence with* X, Z *indecomposables in* $\mathcal{T}_n$. *Then* Y *has at most two indecomposable summands.*

Proof: Let $Y = Y_1 \bigoplus \cdots \bigoplus Y_n$, $n \geq 3$ be the decomposition of Y into indecomposable modules. Since Z is uniserial and $g : Y \to Z$ is an epimorphism, then at least one of g_i's ($g_i : Y_i \to Z$, $i = 1, \ldots, n$) is an epimorphism, say g_1. Consider the following diagram:

$$\begin{tikzcd}
 & Y_1 \arrow[dr, "g_1"] & \\
X \arrow[ur, "f_1"] \arrow[dr, "\tilde{f}"'] & & Z, \\
 & \tilde{Y} \arrow[ur, "-\tilde{g}"'] &
\end{tikzcd}$$

where $\tilde{Y} = Y_2 \oplus \cdots \oplus Y_n$. The sequence ζ is short exact hence the square above is both push-out and pull-back. By the property of the pull-back, we have that $\tilde{f}$ is an epimorphism and $\mathrm{Ker}\, g_1 \cong \mathrm{Ker}\, \tilde{f}$. But $\mathrm{Ker}\, g_1$ is indecomposable, then so is $X/\mathrm{Ker}\, \tilde{f} \cong \tilde{Y}$ and hence $n \leqq 2$. □

Remark 2.2.3 Let ζ be as above. Then we have two cases for Y:

(1) Y is indecomposable. Then $\operatorname{Soc}(Y) = \operatorname{Soc}(X)$, $\operatorname{Top}(Y) = \operatorname{Top}(Z)$ and surely $\ell(Y) = \ell(X) + \ell(Z)$.

(2) $Y = Y_1 \oplus Y_2$. Then $\tilde{Y} = Y_2$, $\tilde{f} = f_2$, $\tilde{g} = g_2$. Since $0 = \operatorname{Ker} f_1 \cap \operatorname{Ker} f_2$ and f_2 is not monomorphism (since then f_2 will be an isomorphism and the sequence will split), we have that f_1 is monomorphism and using the push-out property, so is g_2. So in this case we have f_1, g_2 are monomorphisms and f_2, g_1 are epimorphisms and hence $\operatorname{Top}(X) = \operatorname{Top}(Y_2)$ and $\operatorname{Soc}(Y_2) = \operatorname{Soc}(Z)$. We make another conclusion: Given $0 \neq f : X_i \to X_j$, f is neither monomorphism nor epimorphism, then the following short exact sequence is non-split:

$$\begin{tikzcd}
 & & Y_1 \arrow[rd, "g_1"] & & \\
0 \arrow[r] & X_i \arrow[ru, "f_1"] \arrow[rr, dashed, "f"] \arrow[rd, "f_2"'] & & X_j \arrow[r] & 0, \\
 & & \operatorname{Im} f \arrow[ru, "g_2"'] & &
\end{tikzcd}$$

Now, we prove the following lemma.

Lemma 2.2.4 *Let X be indecomposable in $\mathcal{T}_n$. Then $\operatorname{End}_{\mathcal{T}_n}(X) \cong k[x]/(x^{t+1})$, where $t = \lceil \frac{\ell(X)-1}{n} \rceil$.*

Proof: We notice that for any $0 \neq f : X \to X$, which is neither monomorphism nor epimorphism, we have $X \overset{\pi}{\twoheadrightarrow} \operatorname{Im} f \overset{i}{\hookrightarrow} X$ with $\operatorname{Im} f = \operatorname{Soc}(X) = \operatorname{Top}(X)$, which yields a non-split short exact sequence of the form $0 \to X \to \operatorname{Im} f \oplus Y_2 \to X \to 0$. Now, if $\ell(X) \leq n$, then it is straightforward to see that $\operatorname{End}_{\mathcal{T}_n}(X) \cong k$. Now, let $\ell(X) > n$. Since X is uniserial, there are exactly $\ell(X) - 1$ indecomposable modules with length smaller than $\ell(X)$, which have a socle $\operatorname{Soc}(X)$. Since $\mathcal{T}_n$ is n-periodic, then we have $t = \lceil \frac{\ell(X)-1}{n} \rceil$ modules with the same top and socle as X. At last, we notice that for $s = 1, \ldots, t$ we have $\pi_1^s = \pi_s$, where $X \overset{\pi_1}{\twoheadrightarrow} Y_1 \overset{\pi_2}{\twoheadrightarrow} Y_2 \overset{\pi_3}{\twoheadrightarrow} \cdots \overset{\pi_t}{\twoheadrightarrow} Y_t$, which yields immediately $\operatorname{End}_{\mathcal{T}_n}(X) \cong k[x]/(x^{t+1})$. □

An object X in an abelian category $\mathcal{A}$ is called a **point** if $\operatorname{End}_{\mathcal{A}}(X)$ is a division ring. Two objects X, Y in $\mathcal{A}$ are **orthogonal** if $\operatorname{Hom}_{\mathcal{A}}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(Y, X) = 0$. For example any two simple objects in $\mathcal{A}$ are orthogonal. A sequence $E = (X_1, \ldots, X_k)$ is called an **orthogonal sequence** if any pair (X_i, X_j) for $i \neq j$ is orthogonal.

We comment that if X is a point in $\mathcal{T}_n$ with $\ell(X) < n$, then X does not have self-extensions, and hence by [Br1, Lemma 6.3.4], $\operatorname{add}(X)$ is an exact abelian extension closed subcategory of $\mathcal{T}_n$.

Corollary 2.2.5 *In $\mathcal{T}_n$ the points are all indecomposable modules with length less or equal n.*

Let E be a set of pairwise orthogonal points in $\mathcal{A}$. If A is an object of $\mathcal{A}$, then an E-filtration of A is given by a sequence of subobjects

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A,$$

with $A_i/A_{i-1} \in E$ for $1 \leq i \leq n$. We denote by $\mathcal{U}(E)$ the full subcategory of $\mathcal{A}$ consisting of all objects of $\mathcal{A}$ with an E-filtration. The next theorem, due to Ringel, explains why orthogonal sequences are important.

Theorem 2.2.6 *[R1, Theorem 2] Let E be a set of pairwise orthogonal points in $\mathcal{A}$. Then $\mathcal{U}(E)$ is an exact abelian subcategory which is closed under extensions, and the set E is the set of all simple objects in $\mathcal{U}(E)$.*

For any subset $S \in \mathcal{T}_n$, we define Thick(S) to be the smallest thick subcategory of $\mathcal{T}_n$ containing S, and call it the thick subcategory generated by S.

We associate to every thick subcategory $\mathcal{C}$ in $\mathcal{T}_n$ a sequence of indecomposable modules, called **reduced sequence** as follows: For every simple module $T_i, i = 1, \ldots, n$ of $\mathcal{T}_n$ we take an indecomposable module $X_i \in \mathcal{C}$ such that $\operatorname{Soc}(X_i) = T_i$ and X_i has minimal length or $X_i = 0$ if such module does not exist.

Proposition 2.2.7 *Let $E = (X_1, \ldots, X_k)$ be a reduced sequence in $\mathcal{T}_n$. Then E is orthogonal and each X_i is a point.*

Proof: Let $X_i, X_j \in E$ $(i \neq j)$ and $f : X_i \to X_j$ be a non-zero morphism. By construction of E, f can not be a monomorphism. Also, it can not be epimorphism, since $\operatorname{Soc}(\operatorname{Ker} f) = \operatorname{Soc}(X_i)$ and $\operatorname{Ker} f$ has smaller length. Now, f is neither monomorphism nor epimorphism and by lemma 2.2.1 we have a non-split exact sequence: $0 \to X_i \to Y \to X_j \to 0$ with $Y \in \mathcal{C}$, since $\mathcal{C}$ is closed under extensions. By the remark 2.2.3, we have $Y = Y_1 \bigoplus Y_2$ and $Y_2 \hookrightarrow X_j$, which contradicts the minimal choice of Z and hence $f = 0$. Now, assume that $\operatorname{End}_{\mathcal{T}_n}(X_i)$ is not isomorphic to k. Then by proposition 2.2.4 we have that $\ell(X_i) > n$ and hence we have a non-split short exact sequence $0 \to X_i \to Y \to X_i \to 0$, with $Y = Y_1 \oplus \operatorname{Im} f$. Clearly, $\operatorname{Im} f \in \mathcal{C}$ and $\ell(\operatorname{Im} f) < \ell(X_i)$, which shows that for all $X_i \in E$ we have $\ell(X_i) \leq n$ and hence $\operatorname{End}_{\mathcal{T}_n}(X_i) \cong k$. □

Proposition 2.2.8 *In $\mathcal{T}_n$ there is a bijection between orthogonal sequences and thick subcategories.*

Proof: Let $\mathcal{C}$ be an arbitrary thick subcategory and $E = (E_1, \ldots, E_k)$ be its associate reduced sequence. By proposition 2.2.7, we have that E is orthogonal. Obviously, Thick$(E) \subseteq \mathcal{C}$ has reduced sequence E. We claim that Thick$(E) = \mathcal{C}$. To verify this, take $X \in \mathcal{C}$ with minimal length such that $X \notin$ Thick(E) and consider

$$0 \longrightarrow \operatorname{Soc}(X) \longrightarrow X \longrightarrow X/\operatorname{Soc}(X) \longrightarrow 0.$$

Then $\operatorname{Soc}(X) \in \operatorname{Thick}(E)$ and $\ell(X/\operatorname{Soc}(X)) < \ell(X)$ implies $X/\operatorname{Soc}(X) \in \operatorname{Thick}(E)$. Since $\operatorname{Thick}(E)$ is closed under extensions, then X must be in $\operatorname{Thick}(E)$. Thus, we justified that any thick subcategory is uniquely determined by its reduced sequence. Now, we take an arbitrary orthogonal sequence $E' = (E'_1, \dots, E'_k)$ and consider $\operatorname{Thick}(E')$. We claim that the reduced sequence of $\operatorname{Thick}(E')$ is E'. By definition, we have $\operatorname{Hom}_{\mathcal{T}_n}(E'_i, E'_j) = 0$ for $i \neq j$. If we have non-zero extensions among E'_i's, say $0 \to E'_i \to E''_{ij} \to E'_j \to 0$, and $0 \to E'_j \to E''_{jk} \to E'_k \to 0$, then $\operatorname{Thick}(E''_{ij}, E''_{jk}) = \operatorname{Thick}(E'_i, E'_j, E'_k)$, since $\operatorname{Top}(E''_{ij}) = \operatorname{Top}(E'_j)$, $\operatorname{Soc}(E'_j) = \operatorname{Soc}(E''_{jk})$, and hence E'_j would appear in the middle term of the extension E''_{ij} by E''_{jk}. We conclude that it is not possible to obtain an indecomposable module with length smaller than $\ell(E'_i)$ for $i = 1, \dots, k$ in $\operatorname{Thick}(E'_1, \dots, E'_k)$. Hence E' is the reduced sequence of $\operatorname{Thick}(E')$.□

Corollary 2.2.9 *The number of thick subcategories in $\mathcal{T}_n$ is finite.*

Proof: As noticed in proposition 2.2.7, there is no module with length greater than n which belongs to a reduced sequence, since this module has a self-extension and the middle term has a direct summand with smaller length. Since there are finite number of points in $\mathcal{T}_n$, there are finitely many reduced sequences as well as thick subcategories. □

Recall that an abelian category $\mathcal{C}$ is **connected**, if any decomposition $\mathcal{C} = \mathcal{C}_1 \coprod \mathcal{C}_2$ into abelian categories implies $\mathcal{C}_1 = 0$ or $\mathcal{C}_2 = 0$.

Theorem 2.2.10 *Any thick subcategory of $\mathcal{T}_n$ is exact abelian. More precisely, for any connected thick subcategory $\mathcal{C}$ of $\mathcal{T}_n$, $\mathcal{C}$ is either equivalent to $\operatorname{mod} k\Delta_s$ or to a tube of rank s, where $s \leq n$.*

Proof: Take a thick subcategory $\mathcal{C} \subseteq \mathcal{T}_n$ and its reduced sequence E. We show that $\operatorname{Thick}(E) = \mathcal{U}(E)$. Then using the result of Ringel, $\mathcal{U}(E)$ is exact abelian and hense so is $\mathcal{C} = \operatorname{Thick}(E)$. Obviously, $\operatorname{Thick}(E) \subseteq \mathcal{U}(E)$. Now, since $\mathcal{U}(E)$ is uniserial, for the indecomposable object $M \in \mathcal{U}(E)$ we take its composition series in $\mathcal{U}(E)$: $M \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_{t-1} \supseteq M_t = 0$. Consider the short exact sequence $0 \to M_{t-1} \to M_{t-2} \to M_{t-2}/M_{t-1} \to 0$. We have that M_{t-1} and M_{t-2}/M_{t-1} are simples, hence are in $\operatorname{Thick}(E)$ and since the latter is closed under extensions, we have that $M_{t-2} \in \operatorname{Thick}(E)$. Using the same argument along the composition series of M, we conclude that $M \in \operatorname{Thick}(E)$. Hence $\mathcal{U}(E) = \operatorname{Thick}(E)$. For the last part of the theorem: Note that since $\mathcal{C}$ is hereditary, there exists a finite and connected quiver Q, such that $\mathcal{C} \cong \operatorname{mod} kQ$. Moreover $\mathcal{C}$ is uniserial, hence by [AS, Chapter V.3, Theorem 3.2], $\mathcal{C} \cong \operatorname{mod} k\Delta_s$ or $\mathcal{C} \cong \mathcal{T}_s$, for some $s \leq n$. □

Example 2.2.11 The indecomposable modules of $\operatorname{Thick}(T_i[n])$ are $T_i[kn]$ $(k \in \mathbb{N})$. In fact, $\operatorname{Thick}(T_i[n]) \cong \mathcal{T}_1$.

As we have shown, any thick subcategory $\mathcal{C}$ of $\mathcal{T}_n$ is exact abelian and therefore, it is uniquely determined by its simple objects. The simple modules of $\mathcal{C}$ are among the points of $\mathcal{T}$, which have length at most n. Hence they lie in the $n \times n$ "square": a part of the Auslander-Reiten quiver containing the points of $\mathcal{T}_n$.

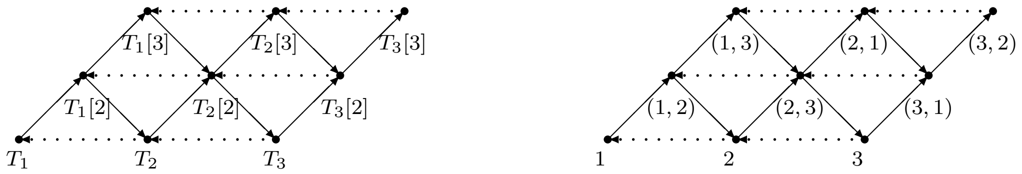


Figure 2.2: The points in $\mathcal{T}_3$

Now, we visualise thick subcategories in $\mathcal{T}_n$ in the following way. We place $1, 2, \ldots, n$ on the circle, which represents the simples of $\mathcal{T}_n$. Since each point is uniquely determined by its socle and its top, we associate to a point in $\mathcal{T}_n$ an **arc** from the circle with **start-point** its socle and **end-point** its top; the direction is clockwise. For a point X, set the length of the arc(X) to be $\ell(X)$ and simply denote the arc(X) by the ordered couple (s, ℓ), where $\mathrm{Soc}(X) = s$ and $\ell(X) = \ell$. The simple objects T_k are represented by the singleton (k). We call two arcs **non-crossing**, if they do not intersect.

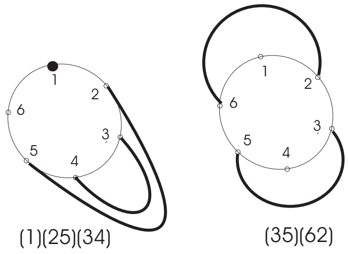


Figure 2.3: Non-crossing arcs on the circle

We say that the arcs arc$(X_i)(i \in I)$ **cover** the circle, if each simple module T_i belongs to the union of supp(X_i).

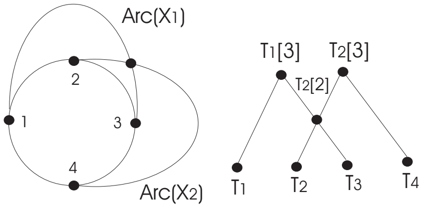
Now, we interpret the morphisms between modules in $\mathcal{T}_n$ in terms of arcs. Let X_1, X_2 be points, $f : X_1 \to X_2$ be a morphism between them and arc(X_1), arc(X_2) be their associated arcs.

(1) If f is a monomorphism, then X_1 and X_2 have the same socle and hence arc(X_1) and arc(X_2) have the same start-point. If f is an epimorphism, then X_1 and X_2 have the same top and hence arc(X_1) and arc(X_2) have the same end-point.

(2) If f is neither monomorphism nor epimorphism, then $\mathrm{Top}(X_1) = \mathrm{Top}(\mathrm{Im}\, f)$, $\mathrm{Soc}(\mathrm{Im}\, f) = \mathrm{Soc}(X_2)$ and hence the arcs intersect. Note that by Happel-Ringel's lemma we have $\mathrm{Ext}^1_{\mathcal{T}_n}(X_2, X_1) \neq 0$.

(3) From (1) and (2) we conclude, that if $f : X_1 \to X_2$ is a non-zero morphism, then the corresponding arcs intersect. We notice that if arc(X_1), arc(X_2) intersect, then $\mathrm{Hom}_{\mathcal{T}_n}(X_1, X_2) \neq 0$ or(and) $\mathrm{Hom}_{\mathcal{T}_n}(X_2, X_1) \neq 0$. The latter is true, since the two arcs intersect in a point Z (algebraic meaning) with $\mathrm{Top}(X_1) = \mathrm{Top}(Z)$ and $\mathrm{Soc}(X_2) = \mathrm{Soc}(Z)$ (or vice versa), and hence we have $0 \neq f : X_1 \twoheadrightarrow Z \hookrightarrow X_2$ (or vice versa). We conclude that $\mathrm{Hom}_{\mathcal{T}_n}(X_1, X_2) = \mathrm{Hom}_{\mathcal{T}_n}(X_2, X_1) = 0$ if and only if the arcs representing these modules are non-crossing.

Example 2.2.12 In $\mathcal{T}_4$ we consider the modules $X_1 = T_1[3]$ and $X_2 = T_2[3]$. Then $\mathrm{arc}(X_1), \mathrm{arc}(X_2)$ intersect, and hence there is a non-zero morphism between $T_1[3]$ and $T_2[3]$, namely $T_1[3] \twoheadrightarrow T_2[2] \hookrightarrow T_2[3]$.



(4) Later we shall use that if $\mathrm{Ext}^1_{\mathcal{T}_n}(X_1, X_2) = \mathrm{Ext}^1_{\mathcal{T}_n}(X_2, X_1) = 0$, then either one of the arc contains the other or there is at least one point between $\mathrm{arc}(X_1), \mathrm{arc}(X_2)$ and at least one point between $\mathrm{arc}(X_2), \mathrm{arc}(X_1)$.

Now, having in mind proposition 2.2.8, we get immediately the following proposition.

Proposition 2.2.13 *There is a bijection between non-crossing arcs on the circle with n points and thick subcategories in $\mathcal{T}_n$.*

Example 2.2.14 In $\mathcal{T}_3$ we consider the thick subcategory $\mathcal{C}_1 = \mathrm{Thick}(T_1[3], T_2)$. Note that $T_1[3]$ and T_2 are simples in $\mathcal{C}$. Then $\mathrm{arc}(T_1[3]) = (1, 3)$ and $\mathrm{arc}(T_2) = (2)$.

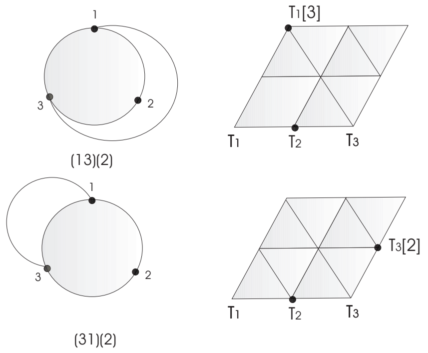


Figure 2.4: Non-crossing arcs and thick subcategories

Now, consider $\mathcal{C}_2 = \mathrm{Thick}(T_3[2], T_2)$. It is easy to see that the simple objects in $\mathcal{C}$ are T_2 and $T_3[2]$. Moreover, $\mathrm{arc}(T_3[2]) = (3, 1)$. Note that different arc orientations, represent different points.

2.3 Cotilting, support-tilting modules and thick subcategories

Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has a **finite generator**, if there is an object $P \in \mathcal{A}$ with $\ell(P) < \infty$ such that for each indecomposable object $X \in \mathcal{A}$ there exist an integer $d \geq 0$ and an epimorphism $P^d \to X$. If the category $\mathcal{A}$ has a finite generator P, we shall write $\mathcal{A} = \mathrm{Gen}(P)$.

We say that $\mathcal{A}$ is **bounded**, if each indecomposable object $X \in \mathcal{A}$ has a bounded length, that is, there is $k \in \mathbb{N}$ such that $\ell(X) < k$. For instance, in $\mathcal{T}_n$ any thick subcategory equivalent to $\mathrm{mod}\, k\Delta_s$, for $s \leq n$ is bounded. If in $\mathcal{A}$ there are indecomposable objects with arbitrary lengths, then we say that $\mathcal{A}$ is **unbounded**. For example, $\mathcal{T}_s$ is unbounded thick for any natural number s.

Recall that the simple composition factors of a module is called the support of the module. For instance, $\mathrm{supp}(T_i[k]) = \{T_i, T_{i+1}, \ldots, T_{i+k-1}\}$, where the indices are taken modulo n and we identify T_0 with T_n. The next lemma elucidates the above notions.

Lemma 2.3.1 *Let $\mathcal{C}$ be a thick subcategory of $\mathcal{T}_n$.*

(i) If the simple objects X_i of $\mathcal{C}$ have pairwise disjoint supports, then $\mathcal{C}$ is bounded if and only if $\sum_{i=1}^{k} \ell(X_i) < n$.

(ii) $\mathcal{C}$ is bounded if and only if $\mathrm{supp}(\mathcal{C}) \subset \{T_1, \ldots, T_n\}$.

Proof: (i) Since X_i's have pairwise disjoint supports, then it is equivalent to say that $\mathrm{arc}(X_i), i = 1, \ldots, k$ do not intersect on the circle, and therefore the sum of the lengths of all these arcs is at most n. Now, suppose that the sum of the length is n, or equivalently that all arcs cover the circle. Then we have a non-split short exact sequence $0 \to X_1 \to Y_1 \to X_2 \to 0$ with Y_1 indecomposable, $\mathrm{Hom}_{\mathcal{T}_n}(Y_1, X_i) = \mathrm{Hom}_{\mathcal{T}_n}(X_i, Y_1) = 0$ for $i = 3, \ldots, k$, $\ell(Y_1) = \ell(X_1) + \ell(X_2)$ and Y_1 belongs to $\mathcal{C}$, since the latter is closed under extensions. Then the sequence $(Y_1, X_3, \ldots, X_k)$ is orthogonal. We apply the same argument for Y_1 and X_3 and following that procedure, at the end we obtain an indecomposable object Y_k with $\mathrm{Soc}(Y_k) = \mathrm{Soc}(X_1)$ and $\ell(Y_k) = \sum_{i=1}^{k} \ell(X_i) = n$, which belongs to $\mathcal{C}$. We conclude that $\mathcal{C}$ is not bounded. Now if $\mathcal{C}$ is unbounded, then there is an indecomposable module X with length $\geq n$, and hence the sum of the lengths of the simples, that appear in the composition series of X (which are among X_i's) is $\geq n$.

(ii) Consider the thick subcategory $\mathcal{C}'$ of $\mathcal{C}$ generated by simples X_i' of $\mathcal{C}$ with $\operatorname{supp}(X_i') \cap \operatorname{supp}(X_j') = \emptyset$ for $i \neq j$. By construction, $\mathcal{C}'$ is obtained from $\mathcal{C}$ by removing a finite number of its bounded thick subcategories. Hence $\mathcal{C}'$ is bounded if and only if $\mathcal{C}$ is bounded. Moreover, $\operatorname{supp}(\mathcal{C}) = \operatorname{supp}(\mathcal{C}')$. Now, $\mathcal{C}'$ satisfies the conditions of (i), hence $\mathcal{C}'$ is bounded if and only if $\sum_{i=1}^{k} \ell(X_i') < n$, which is equivalent to say that $\operatorname{supp}(\mathcal{C}') \subset \{T_1, \ldots, T_n\}$. The claim follows. □

From the proposition follows that $\mathcal{C}$ is unbounded if and only if $\operatorname{supp}(\mathcal{C}) = \{T_1, \ldots, T_n\}$. Then, having in mind proposition 2.2.13, we immediately get the following corollary.

Corollary 2.3.2 *There is a bijection between unbounded thick subcategories in $\mathcal{T}_n$ and non-crossing arcs on the circle with n points that covers the circle.*

For a thick subcategory $\mathcal{C}$ of $\mathcal{T}_n$ we define a new category, namely $\mathcal{C}^{\perp} = \{X \in \mathcal{T}_n \mid \operatorname{Hom}_{\mathcal{T}_n}(\mathcal{C}, X) = \operatorname{Ext}^1_{\mathcal{T}_n}(\mathcal{C}, X) = 0\}$ and call it **right perpendicular of** $\mathcal{C}$. Similarly one defines ${}^{\perp}\mathcal{C} = \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}_n}(X, \mathcal{C}) = \operatorname{Ext}^1_{\mathcal{T}_n}(X, \mathcal{C}) = 0\}$ and call it **left perpendicular of** $\mathcal{C}$. We refer the reader to [GLn] and [Sc] for detailed exposition of perpendicular categories.

The following proposition is from [GLn, Proposition 1.1].

Proposition 2.3.3 *Let $\mathcal{I}$ be a system of objects in an abelian category $\mathcal{A}$. Then the category $\mathcal{I}^{\perp}$ right perpendicular to $\mathcal{I}$ is closed under the formation of kernels and extensions. If additionally,* $\operatorname{proj.dim} \mathcal{I} \leq 1$, *then $\mathcal{I}^{\perp}$ is an exact subcategory of $\mathcal{A}$; i.e., $\mathcal{I}^{\perp}$ is abelian and the inclusion $\mathcal{I}^{\perp} \to \mathcal{A}$ is exact.*

Definition 2.3.4 Let A be a finite dimensional k-algebra. A finitely presented module $T \in \operatorname{mod} A$ is a **partial-tilting** module if

(T1) the projective dimension of T is at most 1;

(T2) $\operatorname{Ext}^1_A(T, T) = 0$.

If additionally,

(T1) there is an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ with each $T_i \in \operatorname{add}(T)$,

then T is called a **tilting** module.

A tilting module is called **basic** if each indecomposable direct summand occurs exactly once in a direct sum decomposition.

In [AS, Chapter VI.4, Corollary 4.4], an alternative characterisation of a tilting module is given.

Proposition 2.3.5 *Let A be a finite dimensional hereditary algebra. A finitely presented module $T \in \operatorname{mod} A$ is a tilting module if*

(T1) $\operatorname{Ext}^1_A(T,T) = 0$*, and*

(T2) the number of pairwise non-isomorphic indecomposable summands of T equals the number of pairwise non-isomorphic simple modules.

A partial tilting A module C is called **support-tilting**, if it is tilting as an $A/\operatorname{ann}(C)$ module. For instance, any simple A module is support-tilting. The following proposition clarifies the notion of a support-tilting module:

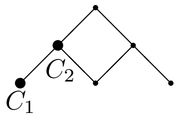
Proposition 2.3.6 *[IT, Proposition 2.5] Suppose that C is a support-tilting A-module. Then the number of distinct indecomposable direct summands of C is the number of distinct simples in its support.*

For the support-tilting modules in $\mathcal{T}_n$, we have the following:

Lemma 2.3.7 *Let C be a support-tilting module. Then* $\operatorname{supp}(C) \subset (T_1, \dots, T_n)$.

Proof: Suppose that $\operatorname{supp}(C) = (T_1, \dots, T_n)$. Then C has n indecomposable direct summands with no extensions between them. But then each indecomposable summand of C has length less then n and hence there is at least two indecomposable summands of C, say C_i, C_j with different socle and top, such that $\operatorname{supp}(C_i) \cap \operatorname{supp}(C_j) \neq \emptyset$. The latter implies that there is an extension between them, see lemma 2.2.2, which is not possible. Hence $\operatorname{supp}(C) \subset (T_1, \dots, T_n)$. □

Example 2.3.8 In $\operatorname{mod} k\Delta_3$ consider the module $C = C_1 \oplus C_2$. The minimal



subquiver on which C is supported is $k\Delta_2$ and C is tilting as a $k\Delta_2$-module. Hence C is a support-tilting module.

We continue with pointing out a relation between support-tilting modules and exact abelian extension closed categories. Let Q be a finite acyclic quiver and kQ be its associated path algebra and $\operatorname{mod} kQ$ is the category of finite dimensional modules over kQ. The following two theorems are from [IT, Section 2.2, 2.3]. We indicate that there the term **wide subcategories** refers to exact abelian extension closed subcategories in our notations.

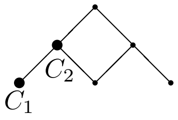
Recall that a **torsion class** is a full subcategory of an abelian category $\mathcal{A}$, which is closed under direct summands, quotients and extensions. We say that an object P in $\mathcal{A}$ is **Ext-projective** if $\operatorname{Ext}^1_{\mathcal{A}}(P, X) = 0$ for any $X \in \mathcal{A}$.

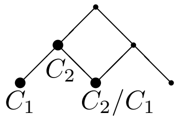
Theorem 2.3.9 *In* $\operatorname{mod} kQ$ *there is a bijection between torsion classes with a finite generator and basic support-tilting modules.*

The bijection is realised as follows:

- Let C be a support-tilting object. Then $\operatorname{Gen}(C)$ is a torsion class having a finite generator.
- Let $\mathcal{C}$ be a torsion class with a finite generator and let C be the direct sum of its indecomposable Ext-projectives. Then C is support-tilting.

Example 2.3.10 We consider again $\operatorname{mod} k\Delta_3$. The module $C = C_1 \oplus C_2$ is support-tilting and $\operatorname{Gen}(C) = \mathcal{U}(C_1, C_2/C_1)$ is exact abelian extension closed. Conversely, the Ext-projectives of $\mathcal{U}(C_1, C_2/C_1)$ are C_1 and C_2.



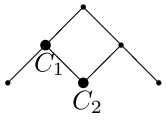


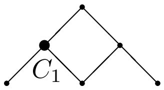
Proposition 2.3.11 *In* $\operatorname{mod} kQ$ *there is a bijection between torsion classes with a finite generator and exact abelian extension closed categories with a finite generator.*

The bijection is given as follows:

- Let $\mathcal{C}$ be a torsion class. Then $\alpha(\mathcal{C}) = \{X \in \mathcal{C} \mid \text{ for all } (g : Y \to X) \in \mathcal{C}, \operatorname{Ker} g \in \mathcal{C}\}$ is exact abelian extension closed.
- If $\mathcal{C}$ is exact abelian extension closed, then $\operatorname{Gen}(\mathcal{C})$ is a torsion class.

Example 2.3.12 We consider again $\operatorname{mod} k\Delta_3$. Then $\mathcal{C} = \operatorname{add}(C_1 \oplus C_2)$ is a torsion class and $\alpha(\mathcal{C}) = \operatorname{add}(C_1)$ is exact abelian extension closed. Conversely, $\operatorname{Gen}(C_1) = \mathcal{C}$.





We make a connection between support-tilting modules and bounded thick subcategories in $\mathcal{T}_n$. We comment that the results discussed so far are not directly applicable to $\mathcal{T}_n$, since the settings are different (the quiver is assumed to be acyclic). Let T be an arbitrary support-tilting module in $\mathcal{T}_n$. Then by proposition 2.3.7, $\operatorname{supp}(T)$ is a proper subset of $\{T_1, \ldots, T_n\}$, say $\operatorname{supp}(T) = \{T_1, \ldots, T_k\}$, for some $k < n$. Then $\operatorname{Thick}(T_1, \ldots, T_k) \cong \operatorname{mod} k\Delta_k$, and since $\operatorname{mod} k\Delta_k$ has a projective generator, then any support-tilting $\mathcal{T}_n$-module is inside some bounded thick subcategory. In that sense, the following theorem is true.

Theorem 2.3.13 *In $\mathcal{T}_n$ there is a bijection between basic support-tilting modules and bounded thick subcategories.*

We enlarge the settings and consider the category of locally nilpotent modules over $k\tilde{\Delta}_n$. The classification of indecomposable objects of category of locally finite modules over $k[[\tilde{\Delta}_n]]$ and hence in $\tilde{\mathcal{T}}_n$ is known, since these categories are equivalent, see [CY, Main Theorem]. Following [BKr, Section 2] we recall it shortly. For each simple object T_i and each $\ell \in \mathbb{N}$ we have a chain of monomorphisms:

$$T_i = T_i[1] \hookrightarrow T_i[2] \hookrightarrow \cdots$$

and denote by $T_i[\infty]$ the **Prüfer module** defined to be $\varinjlim T_i[\ell]$. Note that each Prüfer module is indecomposable injective and $\mathrm{End}_{\tilde{\mathcal{T}}_n}(T_i[\infty]) \cong k[[t]]$.

Lemma 2.3.14 *[BKr, Lemma 2.1] Every non-zero object in $\tilde{\mathcal{T}}_n$ has an indecomposable direct factor, and every indecomposable object is of the form $T_i[\ell]$ for some simple T_i and some $\ell \in \mathbb{N} \cup \{\infty\}$.*

The lemma tells us that $\operatorname{ind}\tilde{\mathcal{T}}_n = \operatorname{ind}\mathcal{T}_n \cup \{\text{Prüfer modules}\}$. This allows us to visualise the Prüfer modules via "extending" the part of the Auslander-Reiten quiver, containing the points in $\mathcal{T}_n$(see figure 2.2) with n extra vertices. In that way, we represent the points in $\tilde{\mathcal{T}}_n$.

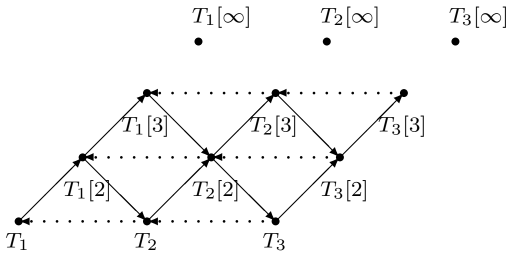


Figure 2.5: The points in $\tilde{\mathcal{T}}_3$

Remark 2.3.15 After knowing the indecomposables in $\tilde{\mathcal{T}}_n$, it is not difficult to show that in $\tilde{\mathcal{T}}_n$, $\mathrm{Thick}(T_i[\infty]) = \mathrm{Thick}(T_i[n])$. To see this, we notice that for $k \in \mathbb{N}$, we have $0 \to T_i[kn] \to T_i[\infty] \overset{\pi_i^k}{\to} T_i[\infty] \to 0$ and since there are no extensions between Prüfer modules, any indecomposable object in $\mathrm{Thick}(T_i[\infty])$ is of the form $T_i[kn]$ for some $k \in \mathbb{N} \cup \{\infty\}$. Now, having in mind that $\varinjlim T_i[k] = \varinjlim T_i[kn]$, the equality above holds. We conclude that simple objects of any thick subcategory in $\tilde{\mathcal{T}}_n$ are among the points of $\mathcal{T}_n$ and hence for any thick subcategory $\tilde{\mathcal{C}}$ of $\tilde{\mathcal{T}}_n$, we have: $\operatorname{ind}\tilde{\mathcal{C}} = \operatorname{ind}\mathcal{C} \cup \{\text{all } T_i[\infty] \mid T_i[n] \in \mathcal{C}\}$, where $\mathcal{C} = \tilde{\mathcal{C}} \cap \mathcal{T}_n$.

Next, we recall the definition of a cotilting object for any abelian category $\mathcal{A}$. To this end, we fix an object T in $\mathcal{A}$. We let $\mathrm{Prod}(T)$ denote the category of all direct summands in any product of copies of T. The object T is called **cotilting** object if the following holds:

(1) the injective dimension of T is at most 1;

(2) $\mathrm{Ext}^1_{\mathcal{A}}(T,T)=0$;

(3) there is an exact sequence $0 \to T_1 \to T_0 \to Q \to 0$ with each T_i in $\mathrm{Prod}(T)$ for some injective cogenerator Q.

In this paper, we shall use an alternative characterisation of a cotilting module, see [BKr, Lemma 1.2], which resembles the one we have for a tilting module.

Proposition 2.3.16 *Let T be an object in $\tilde{\mathcal{T}}_n$ without self-extensions.*

(1) T decomposes uniquely into a coproduct of indecomposable objects having local endomorphism rings.

(2) T is a cotilting object if and only if the number of pairwise non-isomorphic indecomposable direct summands of T equals n.

Now, we recall the following lemma.

Lemma 2.3.17 *[GLn, Lemma 1.2] Let $\mathcal{I}$ and $\mathcal{T}$ be systems of objects of an abelian category $\mathcal{A}$. Then:*

(i) $\mathcal{I} \subset \mathcal{T} \Rightarrow \mathcal{T}^{\perp} \subset \mathcal{I}^{\perp}$.

(ii) $\mathcal{I} \subset^{\perp} (\mathcal{I}^{\perp})$.

(iii) $\mathcal{I}^{\perp} = ({}^{\perp}(\mathcal{I}^{\perp}))^{\perp}$.

For an indecomposable module $X \in \mathcal{T}_n$, set

$$\mathrm{compl}(X) := \{T_i[\infty] \mid \mathrm{Ext}^1_{\mathcal{T}_n}(T_i[\infty], X) = 0\}.$$

We shall use that every morphism between Prüfer objects is an epimorphism, and that there are no morphisms from Prüfer modules to modules in $\mathcal{T}_n$.

Lemma 2.3.18 *Let X, Y be indecomposables with no self-extensions in $\mathcal{T}_n$. Then:*

(i) $\#\{T_i[n] \subseteq X^{\perp}\} = n - \ell(X)$.

(ii) $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y) \Leftrightarrow \{T_i[n] \mid T_i[n] \in Y^{\perp}\} \subseteq \{T_i[n] \mid T_i[n] \in X^{\perp}\}$.

(iii) $\#\{T_i[\infty] \mid \mathrm{Ext}^1_{\mathcal{T}_n}(T_i[\infty], X) \neq 0\} = \#\{\overline{\mathrm{compl}}(X)\} = \ell(X)$.

(iv) $\operatorname{supp}(X) \subseteq \operatorname{supp}(Y) \Leftrightarrow \operatorname{compl}(Y) \subseteq \operatorname{compl}(X)$.

(v) $\operatorname{supp}(X) \cap \operatorname{supp}(Y) = \emptyset \Leftrightarrow \overline{\operatorname{compl}}(X) \cap \overline{\operatorname{compl}}(Y) = \emptyset$.

Proof: After relabeling the simples, we may assume that $X = T_1[\ell]$. Note that both X, Y have lengths $< n$.

(i) Suppose $0 \neq f : T_i[n] \to X$ for some $i \in \{1, \ldots, n\}$. Since $\ell(X) < \ell(T_i[n]) = n$, then f is not a monomorphism. It is immediate to check that if f is an epimorphism, then $\operatorname{Ext}^1_{\mathcal{T}_n}(X, T_i[r]) = 0$ and $\operatorname{Hom}_{\mathcal{T}_n}(X, T_i[n]) = 0$, hence $T_i[n] \in X^{\perp}$, where $i = \ell(X) + 1$. If f is neither monomorphism nor epimorphism, then we have $\operatorname{Ext}^1_{\mathcal{T}_n}(X, T_i[n]) \neq 0$ hence $T_i[n]$ is not in $X^{\perp}$. Therefore if $T_i[n] \in X^{\perp}$ for some $i \in \{1, \ldots, n\}$, then either $\operatorname{Hom}_{\mathcal{T}_n}(T_i[n], X) = 0$ or $\operatorname{Top}(T_i[n]) = \operatorname{Top}(X)$, which is the same to say that $\operatorname{arc}(T_i[n])$ and $\operatorname{arc}(X)$ are either non-crossing or have the same end-point. Hence the number of $T_i[n]$, which are in $X^{\perp}$ equals $n - \ell(X)$. Later we shall use that the indices of all $T_i[n] \subseteq X^{\perp}$ are from the set $i \in I = \{\ell(X)+1, \ell(X)+2, \ldots, n-1, n\}$, and we shall visualise this set as an arc on the circle with consequent integral points.

(ii) Now, since $\ell(X) \leq \ell(Y)$, then due to (i), the indices i for which $\{T_i[n] \subseteq X^{\perp}\}$ are from the set $I = \{\ell(X)+1, \ell(X)+2, \ldots, n-1, n\}$, which obviously contains the set $I' = \{\ell(Y)+1, \ell(Y)+2, \ldots, n-1, n\}$. Since I' is the index set of all i's such that $\{T_i[n] \subseteq Y^{\perp}\}$, the proof follows.

(iii) Let $0 \neq f : T_1[\ell] \to T_i[\infty]$. If f is a mono, then $\operatorname{Ext}^1_{\mathcal{T}_n}(T_{\ell+1}[\infty], X) \neq 0$ since $0 \to T_1[\ell] \to T_1[\infty] \to T_{\ell+1}[\infty] \to 0$ is a non-split exact sequence. Now, exactly as in remark 2.2.2 (ii), any proper epimorphism $f : T_1[\ell] \twoheadrightarrow T_k[\ell+1-k], k = 2, \ldots, \ell$ yields a non-split short exact sequence

$$0 \to T_1[\ell] \to T_1[\infty] \oplus T_k[\ell+1-k] \to T_k[\infty] \to 0,$$

and hence a Prüfer module $T_k[\infty]$ with $\operatorname{Ext}^1_{\mathcal{T}_n}(T_k[\infty], X) \neq 0$. It is straightforward to check that the other Prüfer modules do not have extensions with X. Therefore

$\#\{T_i[\infty] \mid \operatorname{Ext}^1_{\mathcal{T}_n}(T_i[\infty], X) \neq 0\} = \#\{Y \in \mathcal{T}_n \mid \operatorname{Top}(Y) = \operatorname{Top}(X)\} + 1 = (\ell(X) - 1) + 1 = \ell(X)$.

(iv) From (iii) we have that $\overline{\operatorname{compl}}(X) = \{T_2[\infty], T_3[\infty], \ldots, T_\ell[\infty], T_{\ell+1}[\infty]\}$ and hence $\operatorname{compl}(X) = \{T_{\ell+2}[\infty], T_{\ell+3}[\infty], \ldots, T_n[\infty], T_1[\infty]\}$. Since $\operatorname{supp}(X) = \{T_1, T_2, \ldots, T_\ell\}$, we notice that the indices of Prüfer modules in $\overline{\operatorname{compl}}(X)$ are shifted by one (modulo n) the indices of simples in $\operatorname{supp}(X)$. Then $\operatorname{supp}(X) \subseteq \operatorname{supp}(Y) \Leftrightarrow \overline{\operatorname{compl}}(X) \subseteq \overline{\operatorname{compl}}(Y) \Leftrightarrow \operatorname{compl}(Y) \subseteq \operatorname{compl}(X)$.

(v) Follows immediately from (iv). $\square$

Now, we are able to prove the following theorem.

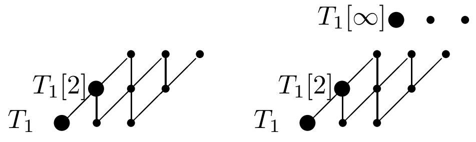
Theorem 2.3.19 *In* $\tilde{\mathcal{T}}_n$ *there is a bijection between cotilting modules and support-tilting modules.*

Proof: Recall that a module X^* is cotilting if and only if it has n indecomposable summands and has no self-extensions. Note that every cotilting module has at least one direct summand which is Prüfer module, since otherwise, we would have that T^* is support-tilting with $\operatorname{supp}(X^*) = (T_1, \ldots, T_n)$, which is impossible, see lemma 2.3.7.

Let $X = \oplus_{i=1}^{t} X_i$ be a support-tilting module having $t < n$ indecomposable summands. First we show that X can be completed by Prüfer modules in a unique way to a cotilting module. The statement will follow at once if we show that there are exactly $n-t$ Prüfer modules in $\operatorname{compl}(X)$. The quiver, on which X is supported, is a disjoint union of k quivers $(1 \leq k < n)$ of type Δ_{s_i} $(i = 1, \ldots, k)$ and since X is support-tilting, we have $\sum_{i=1}^{k} s_i = t$. Then $\operatorname{Thick}(X)$ is a disjoint union of categories of type $\mathcal{C}_i = \operatorname{mod} k\Delta_{s_i}$. Take $X^* = \oplus_{i=1}^{k} X_i^*$ to be a submodule of X such that each X_i^* is indecomposable and $\operatorname{supp}(X_i^*) = \operatorname{supp}(\mathcal{C}_i)$. Then by construction of X^* we have $\operatorname{supp}(X) = \operatorname{supp}(X^*)$ and $\operatorname{supp}(X_i^*) \cap \operatorname{supp}(X_j^*) = \emptyset$ $(i \neq j)$. Now, for appropriate i and j, we have $\operatorname{supp}(X_j) \subseteq \operatorname{supp}(X_i^*)$ and having in mind property (iv), we get $\operatorname{compl}(X) = \cap_{j=1}^{t} \operatorname{compl}(X_j) = \cap_{i=1}^{k} \operatorname{compl}(X_i^*) = \operatorname{compl}(X^*)$. Now, taking into account that $\overline{\operatorname{compl}}(X_j^*) \cap \overline{\operatorname{compl}}(X_i^*) = \emptyset$, we have $\#\{\operatorname{compl}(X)\} = \#\{\operatorname{compl}(X^*)\} = n - \#\{\overline{\operatorname{compl}}(X^*)\} = n - \sum_{i=1}^{k} \#\{\overline{\operatorname{compl}}(X_i^*)\} = n - \sum_{i=1}^{k} \ell(X_i^*)$ $= n - \sum_{i=1}^{k} s_i = n - t$.

Let $Y^* = Y_1 \oplus \cdots \oplus Y_k \oplus Y_{k+1} \oplus \cdots \oplus Y_n$ be a cotilting module and $Y = Y_1 \oplus \cdots \oplus Y_k$ be a submodule of Y such that $Y_1, \ldots, Y_k$ are in $\mathcal{T}_n$ and $Y_{k+1}, \ldots, Y_n$ are Prüfer modules. We show that Y is support-tilting. First we have that Y has no self-extensions. Then it is sufficient to show that the number of simple modules of $\operatorname{supp}(Y)$ is k. Now, since Y has k summands, we have $\#\{\operatorname{supp}(Y)\} \geq k$. But if $\#\{\operatorname{supp}(Y)\} > k$, then $n - k = \#\{\operatorname{compl}(Y)\} = n - \#\{\overline{\operatorname{compl}}(Y)\} = n - \#\{\operatorname{supp}(Y)\} < n - k$, which is impossible. Hence $\#\{\operatorname{supp}(Y)\} = k$ and Y is a support-tilting module. □

Example 2.3.20 Consider the support-tilting module $X = T_1 \oplus T_1[2]$ in $\mathcal{T}_3$ from example 2.3.10. Then $\operatorname{supp}(X) = \{T_1, T_2\}$, $\operatorname{Thick}(T_1, T_2) \cong \operatorname{mod} k\Delta_2 = \mathcal{C}_1$. Now, take $X^* = T_1[2]$. Then $\operatorname{supp}(X^*) = \operatorname{supp}(\mathcal{C}_1)$, $\#\{\operatorname{compl}(X^*)\} = 3 - \ell(X^*) = 3 - 2 = 1$ and $\operatorname{compl}(T_1[2]) = \{T_1[\infty]\}$. Hence the module $T_1 \oplus T_1[2] \oplus T_1[\infty]$ is cotilting.



We return to thick subcategories in $\mathcal{T}_n$. The following proposition relates bounded and unbounded thick categories.

Proposition 2.3.21 *Let $\mathcal{C}$ be a thick subcategory in $\mathcal{T}_n$. If $\mathcal{C}$ is bounded (resp. unbounded), then $\mathcal{C}^\perp$ is unbounded (resp. bounded).*

Proof: Let $\mathcal{C}$ be bounded. First we assume that for the set of simples $\{X_1, X_2, \ldots, X_k\}$ of $\mathcal{C}$ we have $\operatorname{supp}(X_i) \cap \operatorname{supp}(X_j) = 0$ for $i \neq j$. We show that there is a module $X \in \mathcal{C}^\perp$ with $\ell(X) \geq n$, which implies that $\mathcal{C}^\perp$ is unbounded. By lemma 2.3.18(i), we have that for each X_i with $\ell(X_i) = k_i < n$ there are $n - k_i$ modules of length n in $X_i^\perp$. From the same lemma we have that, if, say $\operatorname{arc}(X_1) = (1, \ell)$, then the consequent integral points on the circle $(\ell + 1, \ldots, n)$, which could be interpreted as an arc, represent the indecomposable modules with length n, which are in $X_1^\perp$. We call such an arc an **integral-arc** and for a module X_i, we denote it by A_{X_i}.

Now, disjoint supports of the simples implies that $\sum_{i=1}^k \ell(X_i) < n$, hence any two integral-arcs intersect and cover the circle, since $\ell(A_{X_i}) + \ell(A_{X_j}) = n - \ell(X_i) + n - \ell(X_i) > n$. Moreover, it is not possible that one integral-arc to be contained in other, say $A_{X_j} \subset A_{X_i}$, since this would imply that $\operatorname{supp}(X_i) \subset \operatorname{supp}(X_j)$, which is impossible. Then all such integral-arcs have a non-zero intersection and therefore, there is an indecomposable module of length n which is in $\mathcal{C}^\perp$.

Now, if the supports of the simples of $\mathcal{C}$ are not disjoint, then we form a thick subcategory $\mathcal{C}^* \subseteq \mathcal{C}$ as in lemma 2.3.1(ii) with $\operatorname{supp}(\mathcal{C}^*) = \operatorname{supp}(\mathcal{C})$ and with the property that the simples of $\mathcal{C}^*$ have disjoint supports. Then from the discussions above follows that there is a module of length n which is in $(\mathcal{C}^*)^\perp$. Now, if X_i is a simple module of $\mathcal{C}$, which is not in $\mathcal{C}^*$, then by the construction of $\mathcal{C}^*$ there is a simple module X_i^* of $\mathcal{C}^*$ such that $\operatorname{supp}(X_i) \subseteq \operatorname{supp}(X_i^*)$. Using lemma 2.3.18(ii), any indecomposable module with length n, which is in $(X_i^*)^\perp$, is in $(X_i)^\perp$. The last argument implies that the intersection $\mathcal{C}^* \cap \mathcal{C}$ contains an indecomposable module with length n.

Let $\mathcal{C}$ be an unbounded thick. Take a module $X \in \mathcal{C}$ with $\ell(X) \geq n$. Since there is no module Y in $\mathcal{T}_n$ with $\ell(Y) \geq n$ such that $\operatorname{Ext}^1_{\mathcal{T}_n}(X, Y) = 0$, then $\mathcal{C}^\perp$ has no indecomposable modules of length greater then n and thus, it is bounded. ☐

Remark 2.3.22 In the same way, one can show that forming the left perpendicular category transforms bounded to unbounded thick subcategories and vice versa.

For a thick subcategory $\mathcal{C} \in \mathcal{T}_n$ define $\tau^k\mathcal{C}$ $(k \in \mathbb{Z})$ to be the full subcategory of $\mathcal{T}_n$ whose indecomposable objects are the τ^k-shifts of the indecomposables of $\mathcal{C}$. Also, set $\mathcal{C}^{\perp^0} := \mathcal{C}$ and define inductively $\mathcal{C}^{\perp^k} = (\mathcal{C}^{\perp^{k-1}})^\perp$ if $k > 0$ and $\mathcal{C}^{\perp^k} = {}^\perp(\mathcal{C}^{\perp^{k+1}})$ if $k < 0$.

Before we prove the next proposition, we restate [CB1, Lemma 5].

Lemma 2.3.23 *Let Q be a finite and an acyclic quiver with n vertices and let $\mathcal{C}$ be an exact abelian extension closed subcategory of* $\operatorname{mod} kQ$. *Then* $\operatorname{rk}(\mathcal{C}) + \operatorname{rk}(\mathcal{C}^\perp) = n$.

Proposition 2.3.24 *Let $\mathcal{C} \subseteq \mathcal{T}_n$ be a thick subcategory. Then:*

(i) $\mathrm{rk}(\mathcal{C}^{\perp}) + \mathrm{rk}(\mathcal{C}) = n$.

(ii) ${}^{\perp}(\mathcal{C}^{\perp}) = ({}^{\perp}\mathcal{C})^{\perp} = \mathcal{C}$.

(iii) $\mathcal{C}^{\perp} = {}^{\perp}\tau\mathcal{C}$.

(iv) $\mathcal{C}^{\perp^k} = \mathcal{C}$ *for some* $k \in \mathbb{N}$.

Proof: (i) By proposition 2.3.21, we have that either $\mathcal{C}$ or $\mathcal{C}^{\perp}$ is bounded, so we may assume that $\mathcal{C}$ is bounded. Without loss of generality we also may assume that $\mathcal{C} \subseteq k\Delta_{n-1}$. Denote by $\mathcal{C}^{\perp}_{\Delta_{n-1}} = \mathcal{C}^{\perp} \cap \mathrm{mod}\, k\Delta_{n-1}$. By previous lemma, we have that $\mathrm{rk}(\mathcal{C}^{\perp}_{\Delta_{n-1}}) + \mathrm{rk}(\mathcal{C}) = \mathrm{rk}(\mathrm{mod}\,\Delta_{n-1}) = n-1$. Let X be an indecomposable module with $\mathrm{Soc}(X) = T_n$ and $\ell(X) = n$. Then since $\mathrm{mod}\, k\Delta_{n-1} \subset \mathrm{supp}(X)$, we have $X \in \mathcal{C}^{\perp}$. We claim that $\mathcal{C}^{\perp} = \mathrm{Thick}(\mathcal{C}^{\perp}_{\Delta_{n-1}}, X)$. The inclusion "$\supseteq$" is obvious. Let $S = \mathrm{Thick}(S_1, \ldots, S_k)$ be the set of simples of $\mathcal{C}^{\perp}$. We show that S is contained in $\mathrm{Thick}(\mathcal{C}^{\perp}_{\Delta_{n-1}}, X)$. Suppose $\mathrm{Soc}(E_i) = \mathrm{Soc}(X)$ for some i. Then X/E_i belongs to both $\mathcal{C}^{\perp}$ and $\mathrm{mod}\, k\Delta_{n-1}$ and hence to $\mathcal{C}^{\perp}_{\Delta_{n-1}}$. But then S_i must be in $\mathrm{Thick}(\mathcal{C}^{\perp}_{\Delta_{n-1}}, X)$. If $\mathrm{Soc}(S_i) \in \{T_1, \ldots, T_{n-1}\}$ but $S_i \notin \mathrm{mod}\, k\Delta_{n-1}$, then $\mathrm{supp}(X) \cap \mathrm{supp}(S_i) \neq 0$ and hence $\mathrm{Ext}^1_{\mathcal{T}_n}(S_i, X) \neq 0$. Then one of the middle term is a submodule of S_i and in the same time must be in $\mathcal{C}^{\perp}_{\Delta_{n-1}}$, which contradicts the assumption that S_i is simple. Hence $\mathcal{C}^{\perp} = \mathrm{Thick}(\mathcal{C}^{\perp}_{\Delta_{n-1}}, X)$, which means that $\mathcal{C}^{\perp} = \mathrm{Thick}(S', X)$, where S' is the set of simples of $\mathcal{C}^{\perp}_{\Delta_{n-1}}$. Now, having in mind that $\mathrm{add}(X) \cong \mathcal{T}_1$, then we have $\mathrm{rk}(\mathcal{C}) + \mathrm{rk}(\mathcal{C}^{\perp}) = \mathrm{rk}(\mathcal{C}) + \mathrm{rk}(\mathcal{C}^{\perp}_{\Delta_{n-1}}) + \mathrm{rk}(\mathrm{add}(X)) = \mathrm{rk}(\mathcal{C}) + (n-1) - \mathrm{rk}(\mathcal{C}) + 1 = n$.

(ii) By lemma 2.3.17, we have that $\mathcal{C} \subset {}^{\perp}(\mathcal{C}^{\perp})$. But since $\mathrm{rk}({}^{\perp}(\mathcal{C}^{\perp})) = n - (n - \mathrm{rk}(\mathcal{C})) = \mathrm{rk}(\mathcal{C})$, we have $\mathcal{C} = {}^{\perp}(\mathcal{C}^{\perp})$.

(iii) Using the Auslander-Reiten formula, for X indecomposable in $\mathcal{C}^{\perp}$ we have $0 = \mathrm{Ext}^1_{\mathcal{T}_n}(\mathcal{C}, X) = D\,\mathrm{Ext}^1_{\mathcal{T}_n}(\mathcal{C}, X) = \mathrm{Hom}_{\mathcal{T}_n}(X, \tau\mathcal{C})$ and $0 = \mathrm{Hom}_{\mathcal{T}_n}(\mathcal{C}, X) = \mathrm{Hom}_{\mathcal{T}_n}(\tau\mathcal{C}, \tau X) = D\,\mathrm{Ext}^1_{\mathcal{T}_n}(X, \tau\mathcal{C}) = \mathrm{Ext}^1_{\mathcal{T}_n}(X, \tau\mathcal{C})$.

(iv) Since for every simple module S_i of $\mathcal{C}$ we have $\tau^n S_i = ((S_i^{\perp})^{\perp})^n = S_i^{\perp^{2n}} = S_i$, then applying perpendicular category $2n$ times to $\mathcal{C}$, we return to $\mathcal{C}$. □

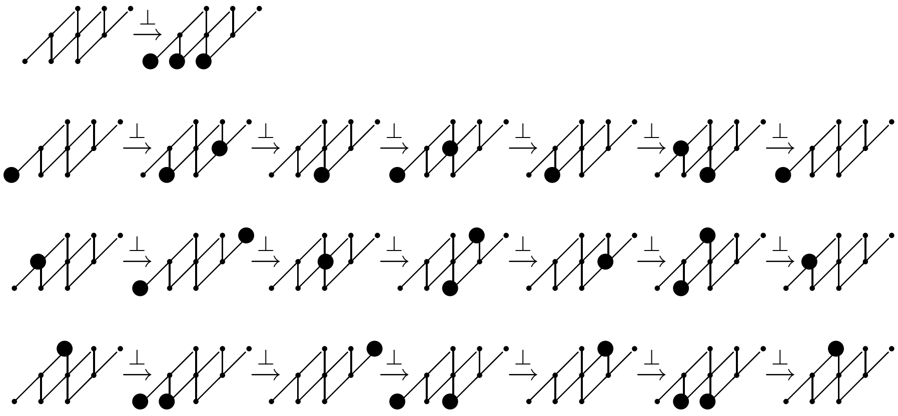
Now, we establish a link between unbounded and bounded thick subcategories.

Theorem 2.3.25 *In $\mathcal{T}_n$ forming the right perpendicular (resp. left perpendicular) category induces a bijection between bounded and unbounded thick subcategories.*

Proof: For an arbitrary bounded (unbounded) thick subcategory $\mathcal{C}$ of $\mathcal{T}_n$, we have that $\mathcal{C}^{\perp}$ is unbounded (bounded). Then using lemma 2.3.24(ii), we have that ${}^{\perp}(\mathcal{C}^{\perp}) = ({}^{\perp}\mathcal{C})^{\perp} = \mathcal{C}$, which yields the bijection. □

Remark 2.3.26 The last theorem gives us an argument, that in $\mathcal{T}_n$, as well as in $\mathrm{mod}\, kQ$ for Q Dynkin quiver, there is $k \in \mathbb{Z}$ such that $\mathcal{C}^{\perp^k} = \mathcal{C}$. In fact, for any hereditary category $\mathcal{C}$, we have ${}^{\perp}(\mathcal{C}^{\perp}) = ({}^{\perp}\mathcal{C})^{\perp} = \mathcal{C}$. Therefore, in both cases right perpendicular is a bijection. Now, having in mind that the number of exact abelian extension closed categories in $\mathcal{T}_n$ and in $\mathrm{mod}\, kQ$ is finite, the claim follows.

Example 2.3.27 The example illustrates lemma 2.3.24(iv) for $n = 3$. The thicken points represent the simples of the respective thick subcategory. Note that $\mathrm{rk}(\mathcal{C}) + \mathrm{rk}(\mathcal{C}^{\perp}) = 3$. We comment that in general, the period does not equals n.



For completeness, we collect all the bijections established so far.

Theorem 2.3.28 *Let $\mathcal{T}_n$ be the category of nilpotent modules and $\tilde{\mathcal{T}}_n$ be the category of locally nilpotent modules over the path algebra $k\tilde{\Delta}_n$. Then there are bijections between:*

(1) support-tilting objects in $\mathcal{T}_n$;

(2) bounded thick subcategories in $\mathcal{T}_n$;

(3) unbounded thick subcategories in $\mathcal{T}_n$;

(4) cotilting objects in $\tilde{\mathcal{T}}_n$.

Proof: Let $X = \oplus_{i=1}^{k} X_i$ be a support-tilting object. Schematically the bijections are described as follows:

$$\begin{tikzcd}
X \arrow[r, bend left, "\alpha(\mathrm{Gen}(X))"] \arrow[d, bend left, "\cap_{i=1}^{k} \mathrm{compl}(X_i)"] & \mathcal{C} \arrow[l, bend left, "\text{Ext-proj. in } \mathrm{Gen}(\mathcal{C})"] \arrow[d, bend left, "|^{\perp}"] \\
X^* \arrow[u, bend left, "\oplus X_i^*{,}\ell(X_i^*)<n"] & \mathcal{C}^{\perp} \arrow[u, bend left, "{}^{\perp}|"]
\end{tikzcd}$$

Theorem 2.3.13 justifies the bijection between (1) and (2).

(1) ⇒ (2). The category $\mathrm{Gen}(X)$ is a torsion class. Then $\alpha(\mathrm{Gen}(X)) = \{Y \in \mathrm{Gen}(X) \mid \text{ for all } (g : Z \to Y) \in \mathrm{Gen}(X), \mathrm{Ker}\, g \in \mathrm{Gen}(X)\}$ is exact abelian extension closed with a finite generator, that is, a bounded thick subcategory in $\mathcal{T}_n$.

(2) ⇒ (1). Given a bounded thick subcategory $\mathcal{C} \subset \mathcal{T}_n$, then $\mathrm{Gen}(\mathcal{C})$ is a torsion class and the direct sum of Ext-projectives in that torsion class is a support-tilting module.

Theorem 2.3.25 establishes bijection between (2) and (3).

(2) ⇔ (3). Let $\mathcal{C}$ be a bounded thick subcategory. Then $\mathcal{C}^{\perp}$(respectively ${}^{\perp}\mathcal{C}$) is unbounded thick and using the perpendicular on the other side, we return to $\mathcal{C}$.

Theorem 2.3.19 yields the bijection between (1) and (4).

(1) ⇒ (4). We complete the support-tilting module X to a cotilting module X^* in a unique way just by taking the intersection of all $\mathrm{compl}(X_i)$.

(4) ⇒ (1). Let $X^* = \oplus_{i=1}^{n} X_i^*$ be a cotilting module. Then the direct sum of all X_i^* such that $\ell(X_i^*) < n$ is a support-tilting module. □

Example 2.3.29 In $\mathcal{T}_3$ consider the support-tilting module $X = T_2[2] \oplus T_3$. Then $X^* = T_2[2] \oplus T_3 \oplus T_2[\infty]$ is a cotilting module in $\tilde{\mathcal{T}}_3$, $\mathcal{C} = \alpha(\mathrm{Gen}(X)) = \mathrm{Thick}(T_2[2])$ is bounded thick and $\mathcal{C}^{\perp} = \mathrm{Thick}(T_1[3], T_2)$ is unbounded thick.

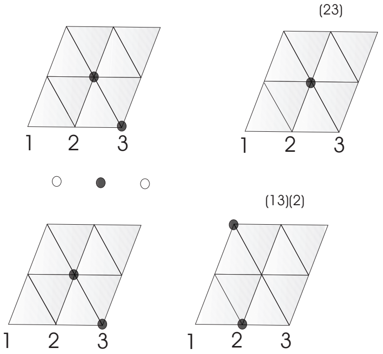


Figure 2.6: Bijections in $\tilde{\mathcal{T}}_3$

At the end of the chapter, we list the rest of bijections in $\tilde{\mathcal{T}}_3$.

2.4 Number of thick subcategories

By a result of Colin Ingalls and Hugh Thomas [IT, Section 3.3], there is a bijection between exact abelian extension closed subcategories with a projective generator in $\mathrm{mod}\, kQ$ and non-crossing partitions of type Q, where Q is a Dynkin or an Euclidean quiver. The number of non-crossing partitions of type Q, where Q is Dynkin quiver is well known. We shall use that when $Q = \Delta_n$ their number is C_{n+1}, where

$C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number, although in the section 2.6, we give another proof.

Let $\mathcal{C}$ be a thick subcategory in $\mathcal{T}_n$ and $E = (S_1, \ldots, S_k)$ be the set of its simple modules. For a simple module T_1 of $\mathcal{T}_n$, define

$$\mathrm{roof}(\mathcal{C}) = \begin{cases} S_i \in E & , \quad T_1 \in \mathrm{supp}(S_i) \text{ and } \ell(S_i) \text{ maximal} \\ 0 & , \quad T_1 \notin \mathrm{supp}(S_i) \text{ for any } S_i \in E \end{cases}$$

$$\mathrm{ht}(\mathcal{C}) = \ell(\mathrm{roof}(\mathcal{C})).$$

Lemma 2.4.1 *Let X be an indecomposable module in $\mathcal{T}_n$ with $\ell(X) = \ell, 1 < \ell \leq n$ and $T_1 \in \mathrm{supp}(X)$.*

(i) $\#\{\mathcal{C} \mid \mathrm{roof}(\mathcal{C}) = X\} = C_{\ell-1}.C_{n-\ell+1}$;

(ii)$\#\{\mathcal{C} \mid \mathrm{ht}(\mathcal{C}) = \ell\} = \ell.C_{\ell-1}.C_{n-\ell+1}$.

Proof: (i) Let $\mathcal{C}$ be a thick subcategory with $\mathrm{roof}(\mathcal{C}) = X$. Without loss of generality we may assume that $\mathrm{Soc}(X) = T_1$. Then $\mathrm{supp}(X) = \{T_1, \ldots, T_\ell\}$ and $\mathrm{Thick}(T_1, \ldots, T_\ell) \cong \mathrm{mod}\, k\Delta_\ell$, if $\ell < n$ or $\mathrm{Thick}(T_1, \ldots, T_\ell) \cong \mathcal{T}_n$, if $\ell = n$. Now, since X is a simple module in $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{T}_n}(X, Y) = \mathrm{Hom}_{\mathcal{T}_n}(Y, X) = 0$, where Y is another simple in $\mathcal{C}$. Hence $\mathrm{supp}(Y) \subsetneqq \{T_2, \ldots, T_{\ell-1}\}$ or $\mathrm{supp}(Y) \subsetneqq \{T_{\ell+1}, \ldots, T_n\}$. Denote by $\mathcal{C}_1 = k\Delta_{\ell-2}$ and $\mathcal{C}_2 = k\Delta_{n-\ell}$ the thick subcategories generated by $\{T_2, \ldots, T_{\ell-1}\}$ and $\{T_{\ell+1}, \ldots, T_n\}$. It is immediate to see that there are neither homomorphism nor extensions between these two categories. Then any thick subcategory $\mathcal{C}$ with $\mathrm{roof}(\mathcal{C}) = X$ must be of the form $\mathcal{C} = \mathrm{Thick}(X, \mathcal{C}^*)$, where $\mathcal{C}^*$ is thick in $\mathcal{C}_1 \oplus \mathcal{C}_2$, see the figure below. But since $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint, then the number of thick subcategories in $\mathcal{C}_1 \oplus \mathcal{C}_2$ is exactly $C_{\ell-1}.C_{n-\ell+1}$.

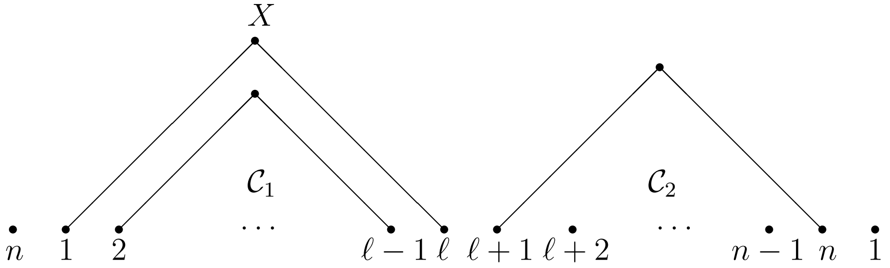


Figure 2.7: Thick subcategories with $\mathrm{roof}(\mathcal{C}) = X$

(ii) The length of X stays invariant under the τ translate hence all thick subcategories $\mathcal{C}_i \in \mathcal{T}_n$ with $\mathrm{ht}(\mathcal{C}_i) = \ell(X)$ are shifts of $\mathcal{C}$, that is, $\mathcal{C}_k = \tau^k(\mathcal{C})$ for appropriate k. Since $\mathrm{roof}(\mathcal{C}_k) = \tau^k(X)$, then $T_1 \in \mathrm{supp}(\tau^k(X))$ if and only if $k = 1, \ldots, \ell(X)$. Hence by (i) $\#\{\mathcal{C} \mid \mathrm{ht}(\mathcal{C}) = \ell\} = \ell.\#\{\mathcal{C} \mid \mathrm{roof}(\mathcal{C}) = X\} = \ell.C_{\ell-1}.C_{n-\ell+1}$. □

Proposition 2.4.2 *The number of thick subcategories in $\mathcal{T}_n$ is $\binom{2n}{n}$.*

Proof: Let $\mathcal{C}$ be a thick subcategory. Since $\mathrm{ht}(\mathcal{C})$ varies from 0 to n, by previous lemma we have: $\#\{\mathcal{C} \in \mathcal{T}_n\} = \sum_{i=0}^{n} \#\{\mathcal{C} \mid \mathrm{ht}(\mathcal{C}) = i\} = C_n + \sum_{i=1}^{n} i.C_{i-1}.C_{n-i+1} =$

$C_n - T_n.C_0 + \sum_{i=1}^{n+1} T_{i-1}.C_{n-i+1} = C_n - T_n + A_n$, where $T_n = (n+1)C_n = \binom{2n}{n}$ and $A_n = \sum_{k=0}^{n} T_k.C_{n-k}$. Here we used that $\{\mathcal{C} \in \mathcal{T}_n \mid \mathrm{ht}(\mathcal{C}) = 0\} = \{\mathcal{C} \in \mathcal{T}_n \mid \mathrm{supp}(\mathcal{C}) = (T_2, \dots, T_n)\} = \{\mathcal{C} \in \mathcal{T}_n \mid \mathcal{C} \in \mathrm{Thick}(T_2, \dots, T_n)\}$ and since $\mathrm{Thick}(T_2, \dots, T_n) \cong \mathrm{mod}\, k\Delta_{n-1}$, we have $\#\{\mathcal{C} \in \mathcal{T}_n \mid \mathrm{ht}(\mathcal{C}) = 0\} = C_n$.

Now, consider the following power series:

$c(x) = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n + \cdots = \sum_{i=0}^{\infty} C_ix^i$, $C_i = \frac{1}{i+1}\binom{2i}{i}$,

$t(x) = T_0 + T_1x + T_2x^2 + \cdots + T_nx^n + \cdots = \sum_{i=0}^{\infty} T_ix^i$, $T_i = \binom{2i}{i}$.

From the Cauchy product of $c(x)$ and $t(x)$ follows that $a(x) = c(x)t(x)$ has $A_n (n = 0, 1, 2, \dots)$ as coefficients, that is,

$a(x) = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n + \cdots = \sum_{i=0}^{\infty} A_ix^i$.

It is a classical result that the power series expansions of $\frac{1-\sqrt{1-4x}}{2x}$ and $\frac{1}{\sqrt{1-4x}}$ are exactly $c(x)$ and $t(x)$.Then

$a(x) = c(x)t(x) = \frac{1}{2x}(\frac{1}{\sqrt{1-4x}} - 1) = \frac{1}{2x}(T_1x + T_2x^2 + \cdots) = \frac{T_1}{2} + \frac{T_2}{2}x + \cdots + \frac{T_{n+1}}{2}x^n + \cdots = \sum_{i=0}^{\infty} A_ix^i$ and comparing the coefficients, we have $A_n = \frac{T_{n+1}}{2}$.

Hence $\#\{\mathcal{C} \in \mathcal{T}_n\} = C_n - T_n + \frac{T_{n+1}}{2} = T_n = \binom{2n}{n}$. □

Corollary 2.4.3 *The number of cotilting modules in* $\tilde{\mathcal{T}}_n$*, support-tilting modules, bounded and unbounded thick subcategories in* $\mathcal{T}_n$ *is* $\binom{2n-1}{n}$.

Proof: By theorem 2.3.28, we have a bijection between bounded, unbounded thick subcategories and support-tilting modules in $\mathcal{T}_n$ and cotilting modules in $\tilde{\mathcal{T}}_n$, hence their number is equal. Now previous proposition tells us that the number of all thick subcategories is $\binom{2n}{n}$ and since the number of bounded thick equals the number of unbounded thick, their number is half of the number of all thick subcategories, that is, $\frac{1}{2}\binom{2n}{n} = \frac{1}{2}\frac{(2n)!}{n!n!} = \frac{1}{2}\frac{(2n-1)!2n}{n!(n-1)!n} = \binom{2n-1}{n}$. □

Remark 2.4.4 In fact, the number of cotilting objects in $\tilde{\mathcal{T}}_n$ is already known, see [BKr, Theorem D].

The following result, first shown by Gabriel, is folklore in the tilting theory. We present another proof by pointing out a connection between certain exact abelian extension closed subcategories and basic tilting modules in $\mathrm{mod}\, k\Delta_n$.

Proposition 2.4.5 *The number of tilting modules in* $\mathrm{mod}\, k\Delta_n$ *is* C_n.

Proof: First we comment that in $\mathrm{mod}\, k\Delta_n$ any thick subcategory is exact abelian. Let $S_1, \dots, S_n$ be the simple and $P_1, \dots, P_n$ be the indecomposable projective $k\Delta_n$-modules. Now, in $\mathrm{mod}\, k\Delta_n$ there is an indecomposable projective-injective module and we denote it by P_n. We shall use again that the number of thick subcategories in $\mathrm{mod}\, k\Delta_n$ is C_{n+1}. First we show that the number of thick subcategories that contain P_n is C_n. Let $\mathcal{C}$ be a thick subcategory with $P_n \in \mathcal{C}$. Then $P_n \subseteq \mathcal{C} \Leftrightarrow \mathcal{C}^{\perp} \subseteq P_n^{\perp}$ and since $\mathcal{C}^{\perp}$ is thick, then the number of thick subcategories of $P_n^{\perp}$ is the same as the number of thick subcategories that contain P_n. But since $P_n^{\perp} = \mathcal{U}(S_1, \ldots, S_{n-1}) = \operatorname{mod} k\Delta_{n-1}$, the claim follows.

Now, we show that there is a bijection between thick subcategories that contain P_n and tilting modules in $\operatorname{mod} k\Delta_n$ and the proof shall follow. From theorem [IT, Theorem 1.1], in $\operatorname{mod} k\Delta_n$ we have a bijection between thick subcategories and support-tilting modules. Now, let $\mathcal{C}$ be a thick subcategory containing P_n. Then since $\operatorname{supp}(P_n) = \{S_1, \ldots, S_n\}$, the corresponding support-tilting module is tilting. Conversely, let T be a tilting module. Then we have the following exact sequence: $0 \to A_A \to T'_A \to T''_A \to 0$ with $T', T'' \in \operatorname{add}(T)$ and $A = k\Delta_n$. Since P_n is also injective, it follows that it is a direct summand of T' and hence of T. Now, $\operatorname{Gen}(T)$ is a torsion class and the corresponding thick subcategory $\alpha(\operatorname{Gen}(T))$ (see proposition 2.3.11) contains P_n, since any morphism $f : X \to P_n$ with X indecomposable in $\operatorname{Gen}(T)$ is a monomorphism, hence $\operatorname{Ker} f = 0 \in \operatorname{Gen}(T)$. The proof follows. $\square$

2.5 Lattice of thick subcategories

As we observed, the set of thick subcategories in $\mathcal{T}_n$ is finite. We consider the poset $(L, \leq)$ formed by subsets of the set of thick subcategories in $\mathcal{T}_n$. In fact, it is not difficult to see that $(L, \leq)$ is a lattice: We notice that intersection of any two thick subcategories $\mathcal{C}_1, \mathcal{C}_2$ is again thick, so we have naturally defined meet in L, namely $\mathcal{C}_1 \wedge \mathcal{C}_2 := \mathcal{C}_1 \cap \mathcal{C}_2$. The join of any two thick subcategories is defined to be the meet of all thick subcategories that contains both of them.

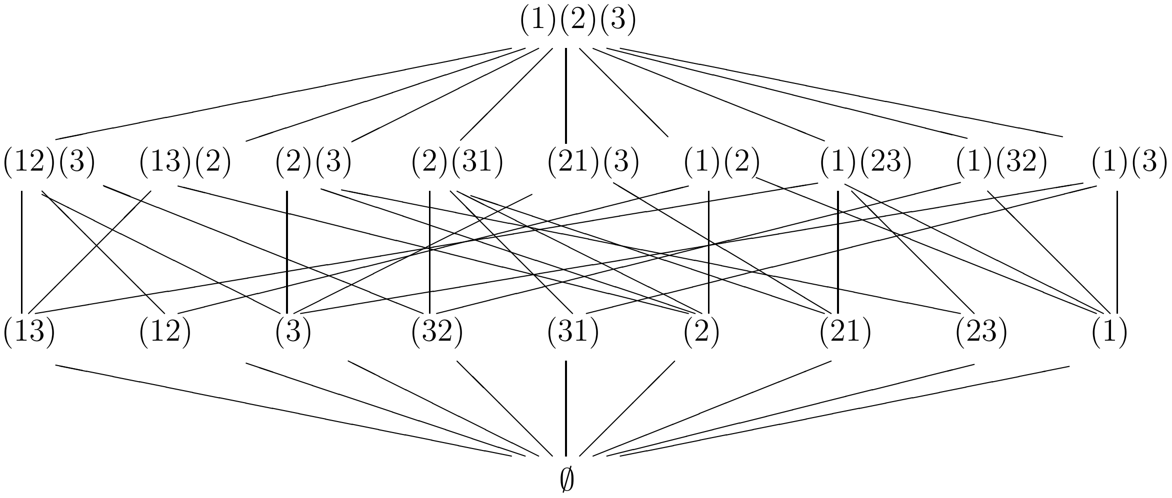


Figure 2.8: The lattice of thick subcategories in $\mathcal{T}_3$

Proposition 2.5.1 *The set of thick subcategories in $\mathcal{T}_n$ forms a lattice. Moreover τ induces a lattice isomorphism and forming the right perpendicular category induces a lattice anti-isomorphism.*

Proof: The first statement follows from the discussions above. For the rest: As we noticed, τ relabels the simples within $\mathcal{T}_n$, hence it yields a lattice isomorphism. If we apply right perpendicular on the set of thick subcategories, then it yields a bijection, see theorem 2.3.25. To show that it is a lattice anti-isomorphism, we check that meets and joints in $(L, \leq)$ are sent to joints and meets in $(L^{\perp}, \leq)$. By lemma 2.3.17, we have that right perpendicular is order reversing, that is, $\mathcal{C}_1 \leq \mathcal{C}_2 \Rightarrow \mathcal{C}_2^{\perp} \leq \mathcal{C}_1^{\perp}$. If $\mathcal{C}_1 = X_1 \vee X_2 \cdots \vee X_k$, where X_i's are the simples of $\mathcal{C}_1$, then we claim that $\mathcal{C}_1^{\perp} = X_1^{\perp} \wedge X_2^{\perp} \cdots \wedge X_k^{\perp}$. First $X_i \leq \mathcal{C}_1$ implies $\mathcal{C}_1^{\perp} \leq X_i^{\perp}$ and hence $\mathcal{C}_1^{\perp} \leq X_1^{\perp} \wedge X_2^{\perp} \cdots \wedge X_k^{\perp}$. If $Y \leq X_1^{\perp} \wedge X_2^{\perp} \cdots \wedge X_k^{\perp}$ is an arbitrary module, then $Y \leq X_i^{\perp}$ for every i and applying left perpendicular we get $X_i \leq {}^{\perp}Y$. Then $\mathcal{C}_1 = X_1 \vee X_2 \cdots \vee X_k \leq {}^{\perp}Y$ and therefore, $Y \leq \mathcal{C}_1^{\perp}$. Since Y was arbitrary, then we get $X_1^{\perp} \wedge X_2^{\perp} \cdots \wedge X_k^{\perp} \leq \mathcal{C}_1^{\perp}$ and the claim follows. We derive that $\mathcal{C} = \mathcal{C}_1 \vee \mathcal{C}_2 \Rightarrow \mathcal{C}^{\perp} = \mathcal{C}_1^{\perp} \wedge \mathcal{C}_2^{\perp}$. In the same way, one shows that for the meet we have $\mathcal{C} = \mathcal{C}_1 \wedge \mathcal{C}_2 \Rightarrow \mathcal{C}^{\perp} = \mathcal{C}_1^{\perp} \vee \mathcal{C}_2^{\perp}$. The proof follows. □

2.6 Nakayama algebras

In this section, we consider certain algebras, which are quotients of $k\Delta_n$ and $k\tilde{\Delta}_n$. We naturally generalise the methods used in the previous sections in order to classify the exact abelian extension closed subcategories for these algebras.

Definition 2.6.1 An algebra A is said to be **left serial** (resp. **right serial**) if every indecomposable projective left (resp. right) A-module is uniserial. It is called **Nakayama algebra** if it is both right and left serial.

We point out that Nakayama algebras are well studied. We recall certain facts for Nakayama algebras, but we refer to [AS, Chapter V] for complete reference to the subject.

Definition 2.6.2 An algebra A is called **basic**, if $e_iA \neq e_jA$ for all $i \neq j$, where $\{e_1, \ldots, e_n\}$ is its complete set of primitive orthogonal idempotents. We say that an algebra A is **connected**, if A is not a direct product of two algebras.

Theorem 2.6.3 *A basic and connected algebra A is a Nakayama algebra if and only if its ordinary quiver Q_A is one of the following quivers:*

(a)
$$\begin{tikzcd} \Delta_n : 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd} ;$$

(b)
$$\begin{tikzcd} \tilde{\Delta}_n : 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend left] \end{tikzcd} .$$

The quotients of Nakayama algebras are again Nakayama.

Proposition 2.6.4 *Let A be an algebra, and J be a proper ideal of A. If A is Nakayama algebra, then A/J is also Nakayama algebra.*

Example 2.6.5 The algebra $k\Delta_n^h = k\Delta_n/I^h$ $(h \geq 1)$, where I is the two-sided ideal generated by all arrows of Δ_n is a Nakayama algebra.

It is not difficult to construct the Auslander-Reiten quiver for the module category over Nakayama algebras. The technique is explained in [AS, Chapter V.4]. We give an example.

Example 2.6.6 The AR-quiver of $\operatorname{mod} k\Delta_6^3$

$$\begin{tikzcd}
 & & S_1[3] \arrow[d] & S_2[3] \arrow[d] & S_3[3] \arrow[d] & S_4[3] \arrow[d] \\
 & S_1[2] \arrow[ur] \arrow[d] & S_2[2] \arrow[l, dotted] \arrow[ur] \arrow[d] & S_3[2] \arrow[l, dotted] \arrow[ur] \arrow[d] & S_4[2] \arrow[l, dotted] \arrow[ur] \arrow[d] & S_5[2] \arrow[l, dotted] \arrow[d] \\
S_1 \arrow[ur] & S_2 \arrow[l, dotted] \arrow[ur] & S_3 \arrow[l, dotted] \arrow[ur] & S_4 \arrow[l, dotted] \arrow[ur] & S_5 \arrow[l, dotted] \arrow[ur] & S_6 \arrow[l, dotted]
\end{tikzcd}$$

We consider the exact abelian extension closed subcategories in $\operatorname{mod} k\Delta_n^h$. Since $\operatorname{mod} k\Delta_n^h \subseteq \operatorname{mod} k\Delta_n$ and $k\Delta_n$ is representation finite, then $k\Delta_n^h$ is also representation finite and hence there are finite number of exact abelian extension closed categories in $\operatorname{mod} k\Delta_n^h$. We denote by $\mathbf{\Delta_n^h}$ their number.

Let A be a Nakayama algebra with simple objects $S_1, \ldots, S_n$, where $n = \operatorname{rk}(\operatorname{mod} A)$. Any exact abelian extension closed category $\mathcal{C}$ in $\operatorname{mod} A$ is uniquely determined by its simple objects S_i^*, that is, $\mathcal{C} = \mathcal{U}(S_1^*, \ldots, S_k^*)$ for $k \leq n$. Since $\mathcal{C} \subseteq \operatorname{mod} A \subseteq \operatorname{mod} k\Delta_n \subset \mathrm{nrep}(\mathrm{k}\tilde{\Delta}_\mathrm{n})$ and the embedding functor is exact, we deduce that in $\operatorname{mod} A$ there is a bijection between orthogonal sequences and exact abelian extension closed categories, as we established for $\mathrm{nrep}(\mathrm{k}\tilde{\Delta}_\mathrm{n})$, see theorem 2.2.8. As we did in the previous section, in order to count the number of exact abelian extension closed subcategories, we count the number of orthogonal sequences in $\operatorname{mod} A$. For a simple module S_1 of $\operatorname{mod} A$, define

$$\operatorname{roof}(\mathcal{C}) = \begin{cases} S_i^* & , \quad S_1 \in \operatorname{supp}(S_i^*) \text{ and } \ell(S_i^*) \text{ maximal} \\ 0 & , \quad S_1 \notin \operatorname{supp}(S_i^*) \text{ for any simple } S_i^* \in E \end{cases}$$

$$\operatorname{ht}(\mathcal{C}) = \ell(\operatorname{roof}(\mathcal{C})).$$

Proposition 2.6.7 *Consider the algebra $k\Delta_n^h$. Then*

$$\mathbf{\Delta_n^h} = \mathbf{\Delta_{n-1}^h} + \sum_{i=0}^{h-1} C_i . \mathbf{\Delta_{n-i-1}^h}, \tag{2.1}$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ is the i^{th} Catalan number.

Proof: Let X be an indecomposable module with $\ell(X) = \ell \geq 2$ and $S_1 \in \mathrm{supp}(X)$. We show that

$$\#\{\mathcal{C} \mid \mathrm{roof}(\mathcal{C}) = X\} = \#\{\mathcal{C} \mid \mathrm{ht}(\mathcal{C}) = \ell\} = \boldsymbol{\Delta}_{\ell-\mathbf{2}}.\boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n}-\ell}. \tag{2.2}$$

For the first equality: By definition X is simple in $\mathcal{C}$ and therefore there is no other, simple module $Y \in \mathcal{C}$ that contains T_1 in its support since this yields $\mathrm{Soc}(X) = \mathrm{Soc}(Y)$ and hence a monomorphism between X and Y, which is impossible. For the second equality, since the simples are orthogonal, from the AR-quiver of $\mathrm{mod}\, k\Delta_n^h$ we notice that all indecomposable objects Y in $\mathrm{mod}\, k\Delta_n^h$ such that $\mathrm{Hom}_{k\Delta_n^h}(X,Y) = \mathrm{Hom}_{k\Delta_n^h}(Y,X) = 0$ are contained in two regions (see the figure below):

- the triangle-shaped region - the part of the AR-quiver that contains all indecomposable objects with support from the set $\{T_2, \ldots, T_{\ell-1}\}$;
- the trapezium-shaped region - the part of the AR-quiver that contains all indecomposable objects with support from the set $\{T_{\ell+1}, \ldots, T_n\}$.

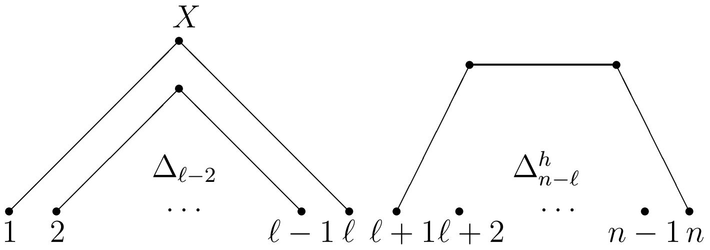


Figure 2.9: The orthogonal points of X

Then $\mathcal{U}(T_2, \ldots, T_{\ell-1}) \cong \mathrm{mod}\, k\Delta_{\ell-2}$, $\mathcal{U}(T_{\ell+1}, \ldots, T_n) \cong \mathrm{mod}\, k\Delta^h_{n-\ell}$ and hence the number of orthogonal sequences in these subcategories is $\boldsymbol{\Delta}_{\ell-\mathbf{2}}$ and $\boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n}-\ell}$ and (2.2) follows at once. If $\ell(X) = 0$ or $\ell(X) = 1$, then evidently all orthogonal to X must be in $\mathcal{U}(T_2, \ldots, T_n) \cong \mathrm{mod}\, \Delta^h_{n-1}$. Now, since $\#\{\mathcal{C} \in \mathrm{mod}\, k\Delta^h_n\} = \sum_{i=0}^{h} \#\{\mathcal{C} \mid \mathrm{ht}(\mathcal{C}) = i\}$, we obtain the following formula:

$$\boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n}} = \boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n-1}} + \boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n-1}} + \sum_{i=2}^{h} \boldsymbol{\Delta}_{\mathbf{i-2}}.\boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n-i}} = \boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n-1}} + \sum_{i=0}^{h-1} \boldsymbol{\Delta}_{\mathbf{i-1}}.\boldsymbol{\Delta}^{\mathbf{h}}_{\mathbf{n-i-1}}, \tag{2.3}$$

where we set $\boldsymbol{\Delta}_{\mathbf{i}} = 1$ for $i < 0$. If $h = n$, then $\mathrm{mod}\, k\Delta^n_n = \mathrm{mod}\, k\Delta_n$ and hence $\boldsymbol{\Delta}^{\mathbf{n}}_{\mathbf{n}} = \boldsymbol{\Delta}_{\mathbf{n}}$ and recurrent formula reads:

$$\boldsymbol{\Delta}_{\mathbf{n}} = \boldsymbol{\Delta}_{\mathbf{n-1}} + \boldsymbol{\Delta}_{\mathbf{n-1}} + \sum_{i=1}^{n-1} \boldsymbol{\Delta}_{\mathbf{i-1}}.\boldsymbol{\Delta}_{\mathbf{n-i-1}}, \tag{2.4}$$

with $\Delta_0 = 1$ and $\Delta_1 = 2$. On the other hand, it is a classical result that for $n \geq 1$ the Catalan numbers are defined via the following recurrent formula:

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} = C_0.C_n + C_0.C_n + \sum_{i=1}^{n-1} C_i C_{n-i}, \tag{2.5}$$

where $C_0 = 1$. Comparing with (2.4), we conclude that $\mathbf{\Delta_i} = C_{i+1}$ and having in mind (2.3), we obtain (2.1). $\square$

Remark 2.6.8 When $h = n$, we obtain that the number of exact abelian extension closed subcategories in $\operatorname{mod} k\Delta_n$ is C_{n+1}. Hence the formula could be interpreted as a generalisation of the recursive formula for the Catalan numbers. For detailed reference to Catalan numbers, we point out [RSt].

We present a table of the number of exact abelian extension closed subcategories in $\operatorname{mod} k\Delta_n^h$. The numbers in bold are the Catalan numbers.

$n\backslash h$	0	1	2	3	4	5	6	7	8
0	**1**	1	1	1	1	1	1	1	1
1	1	**2**	2	2	2	2	2	2	2
2	1	4	**5**	5	5	5	5	5	5
3	1	8	12	**14**	14	14	14	14	14
4	1	16	29	37	**42**	42	42	42	42
5	1	32	70	98	118	**132**	132	132	132
6	1	64	169	261	331	387	**429**	429	429
7	1	128	408	694	934	1130	1298	**1430**	1430
8	1	256	985	1845	2645	3317	3905	4430	**4862**

2.6.1 Self-injective Nakayama algebras

Definition 2.6.9 An algebra A is called **self-injective**, if the left module ${}_AA$ is an injective A-module.

Theorem 2.6.10 *Let A be a basic and connected algebra, which is not isomorphic to k. Then A is a self-injective Nakayama algebra if and only if $A \cong k\tilde{\Delta}_n/R^h$, for some $h \geq 2$, where*

$$\begin{tikzcd}
\tilde{\Delta}_n : 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend left]
\end{tikzcd}$$

with $n \geq 1$ and R is the two-sided ideal generated by all arrows of $\tilde{\Delta}_n$.

Example 2.6.11 The construction of the AR-quiver of $\operatorname{mod} k\tilde{\Delta}_n^h$ is well-known, see [AS, Chapter V.4]. Here is an example for $\operatorname{mod} k\tilde{\Delta}_6^3$.

$$\begin{tikzcd}
 & S_5[3] \arrow[d] & S_6[3] \arrow[d] & S_1[3] \arrow[d] & S_2[3] \arrow[d] & S_3[3] \arrow[d] & S_4[3] \arrow[d] & \\
\cdots & S_6[2] \arrow[d] \arrow[ur] & S_1[2] \arrow[l, dotted] \arrow[d] \arrow[ur] & S_2[2] \arrow[l, dotted] \arrow[d] \arrow[ur] & S_3[2] \arrow[l, dotted] \arrow[d] \arrow[ur] & S_4[2] \arrow[l, dotted] \arrow[d] \arrow[ur] & S_5[2] \arrow[l, dotted] \arrow[d] & \cdots \\
S_6 \arrow[ur] & S_1 \arrow[l, dotted] \arrow[ur] & S_2 \arrow[l, dotted] \arrow[ur] & S_3 \arrow[l, dotted] \arrow[ur] & S_4 \arrow[l, dotted] \arrow[ur] & S_5 \arrow[l, dotted] \arrow[ur] & S_6 \arrow[l, dotted] &
\end{tikzcd}$$

We classify the exact abelian extension closed categories in $\operatorname{mod} k\tilde{\Delta}_n^h$. First we recall that in $\mathrm{nrep}(\mathrm{k}\tilde{\Delta}_{\mathrm{n}})$ the points are all indecomposable modules with length less or equal n. Now, since $\operatorname{mod} k\tilde{\Delta}_n^h \subseteq \mathrm{nrep}(\mathrm{k}\tilde{\Delta}_{\mathrm{n}})$ is a full embedding, the points in $\operatorname{mod} k\tilde{\Delta}_n^h$ are all indecomposables X with $\ell(X) \leq k = \min\{n, h\}$. Since each exact abelian extension closed subcategory of $\operatorname{mod} k\tilde{\Delta}_n^h$ is uniquely determined by its simple objects, which are points, we conclude that all these simples must lie in $\operatorname{mod} \tilde{\Delta}_n^k \subseteq \operatorname{mod} \tilde{\Delta}_n^h$. Having in mind these observations and theorem 2.2.8, together with proposition 2.4.2, we have immediately:

Corollary 2.6.12 *There is a bijection between exact abelian extension closed subcategories of* $\operatorname{mod} k\tilde{\Delta}_n^h$ *and non-crossing arcs with length at most* $k = \min\{n, h\}$ *on a circle with n points. Moreover, the number of exact abelian extension closed subcategories of* $\operatorname{mod} k\tilde{\Delta}_n^h$, *where* $h \geq n$ *is equal to the number of exact abelian extension closed subcategories in* $\mathrm{nrep}(\mathrm{k}\tilde{\Delta}_{\mathrm{n}})$, *which equals* $\binom{2n}{n}$.

Denote by $\tilde{\mathbf{\Delta}}^{\mathbf{h}}_{\mathbf{n}}$ the number of exact abelian extension closed categories in $\operatorname{mod} k\tilde{\Delta}_n^h$.

Proposition 2.6.13 *In* $\operatorname{mod} k\tilde{\Delta}_n^h$ *we have the following recursive formula:*

$$\tilde{\mathbf{\Delta}}^{\mathbf{h}}_{\mathbf{n}} = \mathbf{\Delta}^{\mathbf{h}}_{\mathbf{n-1}} + \sum_{i=1}^{h-1} T_{i-1}.\mathbf{\Delta}^{\mathbf{h}}_{\mathbf{n-i}}, \tag{2.6}$$

where $T_n = \binom{2n}{n}$ *is the central binomial coefficient.*

Proof: Let $\mathcal{C}$ be an exact abelian extension closed subcategory in $\operatorname{mod} \tilde{\Delta}_n^h$ and let X be an indecomposable with $\ell(X) = \ell$. The proof mimics the proof of proposition 2.6.7. The only difference is that $\#\{\mathcal{C} \mid \mathrm{ht}(\mathcal{C}) = \ell\} = \ell.\#\{\mathcal{C} \mid \mathrm{roof}(\mathcal{C}) = X\}$. To verify that, we notice that all exact abelian extension closed subcategories $\mathcal{C}_i$ with $\mathrm{ht}(\mathcal{C}_i) = \ell$ are of the form $\mathcal{C}_i = \tau^i(\mathcal{C})$, $i = 1, \ldots, \ell$. The latter is true since all indecomposable modules with the same length must lie on the same τ-orbit and hence $\mathrm{roof}(\mathcal{C}_i) = \tau^i(\mathrm{roof}(\mathcal{C}))$ and $\mathrm{ht}(\mathcal{C}) = \mathrm{ht}(\mathcal{C}_k)$. It is exactly ℓ-times

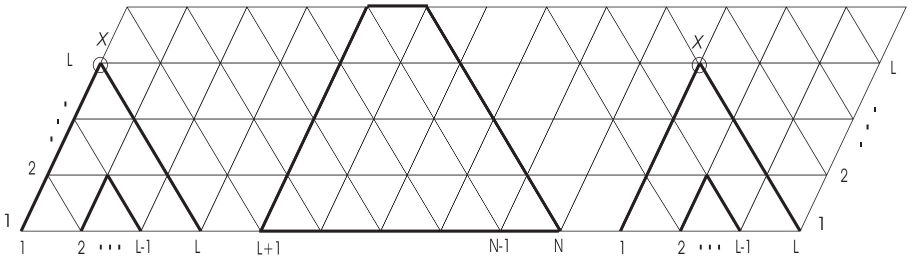


Figure 2.10: ℓ-times shifts of thick subcategories

since $X_1 \in \mathrm{supp}(\tau^i X)$ for $i = 1, \dots, \ell$. We conclude that $\#\{\mathcal{C} \mid \mathrm{ht}(\mathcal{C}) = \ell\} = \ell.C_{\ell-1}.\mathbf{\Delta^h_{n-i}} = T_\ell.\mathbf{\Delta^h_{n-i}}$. □

Here is a table of the number of exact abelian extension closed subcategories in $\mathrm{mod}\, k\tilde{\Delta}^h_n$. The numbers in bold are the central binomial coefficients.

$n\backslash h$	0	1	2	3	4	5	6	7	8
0	**1**	1	1	1	1	1	1	1	1
1	1	**2**	2	2	2	2	2	2	2
2	1	4	**6**	6	6	6	6	6	6
3	1	8	14	**20**	20	20	20	20	20
4	1	16	34	50	**70**	70	70	70	70
5	1	32	82	132	182	**252**	252	252	252
6	1	64	198	354	504	672	**924**	924	924
7	1	128	478	940	1430	1920	2508	**3842**	3432
8	1	256	1154	2498	4078	5646	7326	9438	**12870**

Remark 2.6.14 In fact, using similar arguments as in the last proposition, one gets the following recursive formula:

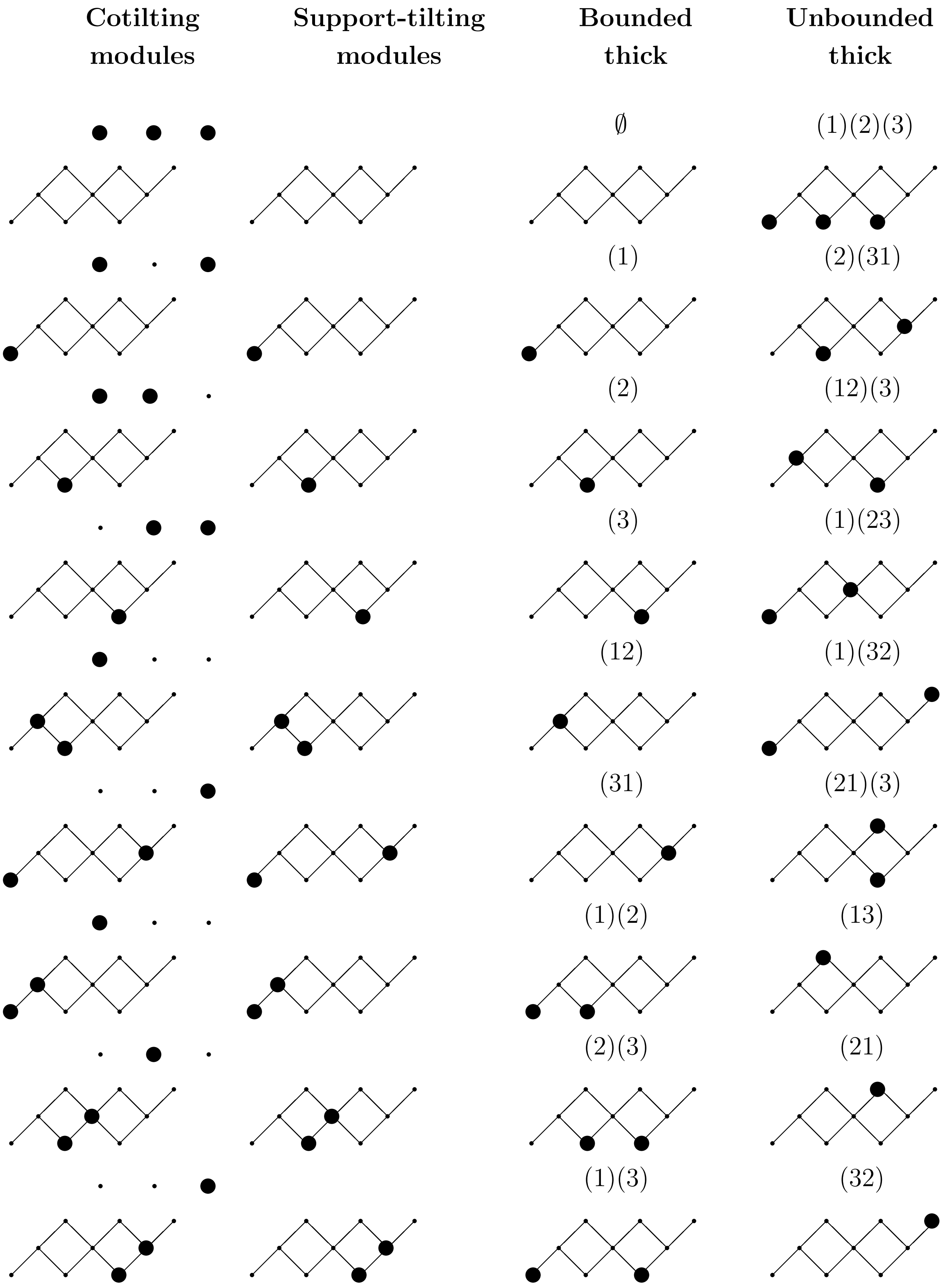
Proposition 2.6.15 *Consider the algebra $k\tilde{\Delta}^h_n$. Then*

$$\mathbf{\tilde{\Delta}^h_n} = \mathbf{\tilde{\Delta}^h_{n-1}} + \sum_{i=1}^{h-1} C_{i-1}.\mathbf{\tilde{\Delta}^h_{n-i}}, \tag{2.7}$$

where C_n is the n^{th} Catalan number.

If we compare (2.6) and (2.7), we notice that the last formula is more coherent in a sense that it involves terms from the same type.

We illustrate the bijections established in theorem 2.3.28. The thicken points (•) represent the indecomposable direct summands of the cotilting and support-tilting modules and the simples of the thick subcategories. For simplicity, we do not set labels of the indecomposable modules, but we refer to figure 2.5.



Chapter 3

Thick subcategories for hereditary algebras

For a finite and acyclic quiver Q, we consider its path algebra kQ. We step on a result of Crawley-Boevey [CB1, Lemma 5], which says that any thick subcategory of $\operatorname{mod} kQ$ generated by an exceptional sequence is exact abelian. We construct for a thick subcategory $\mathcal{C} \subseteq \operatorname{mod} kQ$ generated by preprojective modules, an exceptional sequence that generates $\mathcal{C}$.

Next, we specialise to the module category of kQ, where Q is an Euclidian quiver. Its path algebra is an example of representation-infinite hereditary algebra, for which the classification of indecomposable modules is well-known. We introduce reduction techniques, some of which work in a more general settings, which enable us to prove that any thick subcategory in $\operatorname{mod} kQ$ is exact abelian.

By a result of Colin Ingalls and Hugh Thomas [IT, Theorem 1.1], there is a bijective correspondence between non-crossing partitions associated to Q (Q is an Euclidian quiver) and exact abelian extension closed subcategories with a projective generator in $\operatorname{mod} kQ$. As one observes, there are exact abelian extension closed subcategories without a projective generator (for instance the tubes in the regular component of the Auslander-Reiten quiver of $\operatorname{mod} kQ$). So we use results from the second chapter, and combining with the above cited theorem, we give a complete combinatorial classification of thick subcategories in $\operatorname{mod} kQ$.

The results in this chapter are joint work with Yu Ye.

3.1 Thick subcategories generated by preprojective modules

From now on, we assume that Q is a finite and acyclic quiver and k an is algebraically closed field. We begin with recalling some facts for the structure of the Auslander-

Reiten quiver of mod kQ. As a reference, we point out [AS, Chapter VIII.2].

Definition 3.1.1 Let A be an arbitrary (not necessarily hereditary) k-algebra, and $\Gamma(\operatorname{mod} A)$ the Auslander-Reiten quiver of A.

(a) A connected component $\mathcal{P}$ of $\Gamma(\operatorname{mod} A)$ is called **preprojective** if $\mathcal{P}$ is acyclic and, for any indecomposable module M in $\mathcal{P}$, there exist $t \geq 0$ and $a \in (Q_A)_0$ such that $M \cong \tau^{-t}P(a)$. An indecomposable A-module is called **preprojective** if it belongs to a preprojective component of $\Gamma(\operatorname{mod} A)$, and an arbitrary A-module is called **preprojective** if it is a direct sum of indecomposable preprojective A-modules.

(b) A connected component $\mathcal{Q}$ of $\Gamma(\operatorname{mod} A)$ is called **preinjective** if $\mathcal{Q}$ is acyclic and, for any indecomposable module M in $\mathcal{Q}$, there exist $s \geq 0$ and $b \in (Q_A)_0$ such that $M \cong \tau^{s}I(b)$. An indecomposable A-module is called **preinjective** if it belongs to a preinjective component of $\Gamma(\operatorname{mod} A)$, and an arbitrary A-module is called **preinjective** if it is a direct sum of indecomposable preinjective A-modules.

Proposition 3.1.2 *Let Q be a finite, connected, and acyclic quiver, and let $A = kQ$. The quiver $\Gamma(\operatorname{mod} A)$ contains a preprojective $\mathcal{P}(A)$ and preinjective $\mathcal{Q}(A)$ component.*

Now we look at the structure of the preprojective (preinjective) component of $\Gamma(\operatorname{mod} A)$. Let M, N be two indecomposable A-modules. A **path** in mod A from M to N of length t is a sequence:

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_t} M_t = N$$

where all the M_i are indecomposable, and all f_i are non-zero nonisomorphisms. In this case, M is called a **predecessor** of N in mod A. Dually, one has a definition of a **successor**. We have the following proposition.

Proposition 3.1.3 *[AS, Chapter VIII.2, Proposition 2.1] Let A be arbitrary (not necessarily hereditary) algebra.*

(a) Let $\mathcal{P}$ be a preprojective component of the quiver $\Gamma(\operatorname{mod} A)$ and M be an indecomposable module in $\mathcal{P}$. Then the number of predecessors of M in $\mathcal{P}$ is finite and any indecomposable A-module L such that $\operatorname{Hom}_A(L, M) \neq 0$ is a predecessor of M in $\mathcal{P}$. In particular, $\operatorname{Hom}_A(L, M) = 0$ for all but finitely many indecomposable A-modules L.

(b) *Let $\mathcal{Q}$ be a preinjective component of the quiver* $\Gamma(\operatorname{mod} A)$ *and N be an indecomposable module in $\mathcal{Q}$. Then the number of successors of N in $\mathcal{Q}$ is finite and any indecomposable A-module L such that* $\operatorname{Hom}_A(N, L) \neq 0$ *is a successor of N in $\mathcal{Q}$. In particular,* $\operatorname{Hom}_A(N, L) = 0$ *for all but finitely many indecomposable A-modules L.*

A path from an indecomposable A-module to itself, is a sequence on non-zero nonisomorphisms between indecomposables of the form

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_t} M_t = M,$$

is called a **cycle** in $\operatorname{mod} A$. Then the previous proposition says that, in case of modules lying in preprojective or preinjective components, these module-theoretical notions can be expressed graphically.

Proposition 3.1.4 *[AS, Chapter VIII.2, Corollary 2.6] Let A be an arbitrary (not necessarily hereditary) k-algebra.*

(a) *Let $\mathcal{P}$ be preprojective component of* $\Gamma(\operatorname{mod} A)$ *and M be an indecomposable module in $\mathcal{P}$. Then*

  (i) *any predecessor L of M in* $\operatorname{mod} A$ *is preprojective and there is a path in $\mathcal{P}$ from L to M, and*

  (ii) *M lies on no cycle in* $\operatorname{mod} A$.

(b) *Let $\mathcal{Q}$ be preinjective component of* $\Gamma(\operatorname{mod} A)$ *and N be an indecomposable module in $\mathcal{Q}$. Then*

  (i) *any successor N of L in* $\operatorname{mod} A$ *is preinjective and there is a path in $\mathcal{Q}$ from N to L, and*

  (ii) *N lies on no cycle in* $\operatorname{mod} A$.

A kQ-module X is called **exceptional**, provided that X is indecomposable and $\operatorname{Ext}^1_{kQ}(X, X) = 0$. Examples of exceptional modules are the simple modules. The following lemma gives more examples.

Lemma 3.1.5 *[AS, Chapter VIII.2, Lemma 2.7] Let A be an arbitrary (not necessarily hereditary) k-algebra and M be an indecomposable preprojective, or preinjective, A-module. Then* $\operatorname{End}_A M \cong k$ *and* $\operatorname{Ext}^1_A(M, M) = 0$.

Let $E = (X_1, \ldots, X_r)$ be a sequence of kQ-modules. Then E is **exceptional sequence** of length r, if all X_i are exceptional and $\operatorname{Hom}_{kQ}(X_j, X_i) = 0$ for $1 \leq i < j \leq r$ and $\operatorname{Ext}^1_{kQ}(X_j, X_i) = 0$ for $1 \leq i \leq j \leq r$. If r equals the number of vertices of Q, then E is called **complete**. Recall that E is called **orthogonal**, if $\mathrm{Hom}_{kQ}(X_i, X_j) = 0$ for any $i \neq j$. We refer the reader to papers of [R3] and [CB1], where exceptional sequences are studied in great details.

We notice that since Q has no oriented cycles, we can relabel the vertices of Q such that $(S_n, S_{n-1}, \dots, S_1)$ forms an exceptional sequence. In that case, it is immediate to check that the sequence $(P_1, P_2, \dots, P_n)$ of projectives is also exceptional. So, from now on, we assume that we label the vertices of Q in such a way, that the above sequences are exceptional.

As before, we denote by $\mathrm{Thick}(S)$ the smallest thick subcategory containing S, where S is an arbitrary set of kQ-modules. We shall frequently use the following lemma.

Lemma 3.1.6 *[CB1, Lemma 5]) Let E be an exceptional sequence of length r in* $\mathrm{mod}\, kQ$. *Then* $\mathrm{Thick}(E)$ *is equivalent to the category of representations of a quiver $Q(E)$ with r vertices and no oriented cycles. The functor* $\mathrm{mod}\, kQ(E) \hookrightarrow \mathrm{mod}\, kQ$ *is exact and induces isomorphism on both* Hom *and* Ext. *Moreover, any exceptional sequences in* $\mathrm{mod}\, kQ$ *can be enlarged to a complete sequence.*

In other words $\mathrm{Thick}(E)$, for E exceptional, is an exact abelian extension closed subcategory of $\mathrm{mod}\, kQ$.

After recalling these facts, we start with our investigation. Let $\{S_1, \dots, S_n\}$ be the complete set of simple kQ-modules, and $\{P_1, \dots, P_n\}$ and $\{I_1, \dots, I_n\}$ the corresponding indecomposable projective and injective modules.

We consider the preprojective component $\mathcal{P} = \{\tau^m P_i, m \leq 0, 1 \leq i \leq n\}$ of the Auslander-Reiten quiver of $\mathrm{mod}\, kQ$. The structure of $\mathcal{P}$ (see theorem 3.1.3) allows us to introduce a total order on $\mathcal{P}$ as follows: $\tau^m P^i \prec \tau^n P^j$ if $m > n$ or $m = n$ and $i < j$. Obviously, $\mathrm{Hom}_{kQ}(X_1, X_2) = 0$ for any $X_1 \succ X_2$ in $\mathcal{P}$. For any $X_1, X_2 \in \mathcal{P}$, the distance $d(X_1, X_2)$ between X_1 and X_2 is defined to be the supremum of the lengths of paths starting in X_1 and terminating at X_2 in the Auslander-Reiten quiver of kQ, and 0 when no such a path exists.

Example 3.1.7 We consider a part of the preprojective component of $\Gamma(\mathrm{mod}\, kQ)$.

$$\begin{tikzcd}
 & P_2 \arrow[d, dotted, no head] & {} \arrow[r, dotted] & {} & & \tau^{-k}P_2 \arrow[d, dotted, no head] \arrow[dr, "g_1"] & & {} \arrow[r, dotted] & {} \\
P_1 \arrow[ur] \arrow[r] \arrow[dr] & P_k \arrow[d, dotted, no head] & {} \arrow[r, dotted] & {} & \tau^{-k}P_1 \arrow[ur, "f_1"] \arrow[r] \arrow[dr, "f_n"'] & \tau^{-k}P_k \arrow[r] \arrow[d, dotted, no head] & \tau^{-k-1}P_1 & {} \arrow[r, dotted] & {} \\
 & P_n & {} \arrow[r, dotted] & {} & & \tau^{-k}P_n \arrow[ur, "g_n"'] & & {} \arrow[r, dotted] & {}
\end{tikzcd}$$

Now, $d(\tau^{-k}P_1, \tau^{-k}P_i) = 1$, $d(\tau^{-k}P_1, \tau^{-k-1}P_1) = 2$ and $d(\tau^{-k-t}P_i, \tau^{-k}P_1) = 0$ for $k, t \in \mathbb{N}$ and $i = 2, \dots, n$.

The following facts are easily derived from the definition.

Lemma 3.1.8 *Let Q be a quiver and $X, Y \in \mathcal{P}$.*

(i) If $X \succ Y$, then $d(X,Y)$=0.

(ii) If $\mathrm{Hom}_{kQ}(X,Y) \neq 0$ and $X \neq Y$, then $d(X,Y) \geq 1$; if $\mathrm{Ext}^1_{kQ}(Y,X) \neq 0$, then $d(X,Y) \geq 2$.

(iii) For any given $X \in \mathcal{P}$ and $d > 0$, there exist only finitely many $Y \in \mathcal{P}$, such that $0 < d(X,Y) \leq d$ or $0 < d(Y,X) \leq d$.

By induction on distance, we get the following useful lemma.

Lemma 3.1.9 *Let $S \subseteq \mathcal{P}$ be a set of kQ-modules and $Z \in \mathcal{P}$. Then there exists a set $S^* \subseteq \mathcal{P}$, such that $\mathrm{Thick}(S^*, Z) = \mathrm{Thick}(S, Z)$, $\mathrm{Ext}^1_{kQ}(Z,X) = 0$ for any $X \in S^*$, and $\mathrm{Hom}_{kQ}(X,Z) \neq 0$ for any $X \in S^* \setminus S$.*

Proof: Set $d(S;Z) = \sup(\{d(X,Z) \mid X \in S, \mathrm{Ext}^1_{kQ}(Z,X) \neq 0\})$ if $\mathrm{Ext}^1_{kQ}(Z,X) \neq 0$ for some $X \in S$, and 0 otherwise. We use induction on $d(S;Z)$. By lemma 3.1.8, $d(S;Z) = 0$ or $d(S;Z) \geq 2$. If $d(S;Z) = 0$, then $\mathrm{Ext}^1_{kQ}(Z,X) = 0$ for all $X \in S$ and hence we may take $S^* = S$. So, we may assume that $d(S;Z) > 0$.

For any $X \in S$ such that $\mathrm{Ext}^1_{kQ}(Z,X) \neq 0$, we fix a non-split short exact sequence $0 \to X \to \oplus_{i=1}^{l} X_i^{\oplus n_i} \to Z \to 0$, where X_i's are indecomposable and pairwise non-isomorphic. Set $S_{X;Z} = \{X_1, \ldots, X_l\}$. By construction, we have $\mathrm{Thick}(X,Z) = \mathrm{Thick}(S_{X;Z}, Z)$ and $d(S_{X;Z}; Z) \leq d(X;Z) - 1$. The last equality holds since $d(X,Z) \geq d(X,Y) + d(Y,Z)$ for any X, Y and $Z \in \mathcal{P}$, provided that $d(X,Z) \neq 0$, $d(X,Y) \neq 0$ and $d(Y,Z) \neq 0$.

Now, we take

$$S' = \{X \in S \mid \mathrm{Ext}^1_{kQ}(Z,X) = 0\} \cup \bigcup_{X \in S, \mathrm{Ext}^1_{kQ}(Z,X) \neq 0} S_{X;Z}.$$

We showed that $d(S';Z) < d(S;Z)$ and $\mathrm{Thick}(S', Z) = \mathrm{Thick}(S, Z)$. Clearly, $\mathrm{Hom}_{kQ}(X,Z) \neq 0$ for any $X \in S' \setminus S$. Now, repeat the argument for S', and after finite steps we get $S^* \subset \mathcal{P}$ with the desired properties. □

Proposition 3.1.10 *Let $S \subset \mathcal{P}$ be a set of kQ-modules. Then there exists an exceptional sequence E, such that $\mathrm{Thick}(S) = \mathrm{Thick}(E)$. As a consequence, any thick subcategory generated by preprojective modules is exact abelian.*

Proof: We use induction on the total order on $\mathcal{P}$. First we assume that S is a finite set. We construct the required exceptional sequence E in the following way.

We take a maximal element Z_1 in S. This can be done since S is a finite set. Set $S' = S \setminus \{Z_1\}$. Now, we know that $\mathrm{Hom}_{kQ}(Z_1, X) = 0$ for any $X \in S'$. By lemma 3.1.9, there exists $S_1 \subseteq \mathcal{P}$, such that $\mathrm{Thick}(S_1, Z_1) = \mathrm{Thick}(S)$, $\mathrm{Ext}^1_{kQ}(Z_1, X) = 0$ for any $X \in S_1$, and moreover, $\mathrm{Hom}_{kQ}(Z, X) = 0$ for any $X \in S_1$.

In other words, $\mathrm{Thick}(S_1, Z_1) = \mathrm{Thick}(S)$ and $\mathrm{Thick}(S_1) \subseteq Z_1^{\perp}$, where as usual $Z_1^{\perp} = \{X \in \mathrm{mod}\, kQ \mid \mathrm{Hom}_{kQ}(Z_1, X) = \mathrm{Ext}^1_{kQ}(Z_1, X) = 0\}$.

Set Z_1 to be the last term of E and repeat the argument on S_1 to get an ascending sequence $E = \{\dots, Z_2, Z_1\}$ in $\mathcal{P}$ with respect to the order we defined before, such that $\mathrm{Thick}(E) = \mathrm{Thick}(S)$ and $\mathrm{Thick}(\dots, Z_{n+2}, Z_{n+1}) \subseteq {Z_n}^{\perp}$ for any $n \geq 1$. Since for any $Z \in \mathcal{P}$, there exist only finitely many $X \in \mathcal{P}$ with $X \prec Z$, we will stop after finite steps, which means that E is a finite sequence. By construction, E is an exceptional sequence and $\mathrm{Thick}(E) = \mathrm{Thick}(S)$.

Now, let S be an arbitrary subset of $\mathcal{P}$. Since there exist only countably many preprojective modules, we assume that $S = \{X_1, X_2, \dots\}$. Set $S_i = \{X_1, X_2, \dots, X_i\}$ and $\mathcal{C}_i = \mathrm{Thick}(S_i)$ for any $i \geq 1$. We complete the proof by showing that $\mathrm{Thick}(S) = \mathcal{C}_k$ for some k. Otherwise, assume that there exists an ascending sequence $1 = r_1 < r_2 < r_3 < \cdots$, such that

$$\mathcal{C}_1 = \mathcal{C}_{r_1} \underset{\neq}{\subseteq} \mathcal{C}_{r_2} \underset{\neq}{\subseteq} \mathcal{C}_{r_3} \underset{\neq}{\subseteq} \cdots .$$

We showed that each $\mathcal{C}_{r_i} = \mathrm{Thick}(E_{r_i})$, for some exceptional sequence E_{r_i}, and hence $\mathcal{C}_{r_i}$ is isomorphic to the finite dimensional module category of some quiver. Now, fix a complete sequence F_1 in $\mathcal{C}_{r_1}$. The latter can be enlarged to a complete sequence F_2 in $\mathcal{C}_{r_2}$, and do this repetitively to get an exceptional sequence F_i for any i. We know that each F_i is an exceptional sequence in $\mathrm{mod}\, kQ$. Since the length of an exceptional sequence is at most n, we know that there exists k, such that $F_i = F_k$ for any $i \geq k$, which contradicts the assumption that $\mathcal{C}_{r_i} \neq \mathcal{C}_{r_{i+1}}$ for any i. □

Remark 3.1.11 Dually, we can prove that if $S \subseteq \mathcal{Q}$ is a set of kQ-modules, then there exists an exceptional sequence $E \subseteq \mathcal{Q}$ such that $\mathrm{Thick}(E) = \mathrm{Thick}(S)$. Hence any thick subcategory generated by preinjective modules is exact abelian.

Corollary 3.1.12 *Let Q be a Dynkin quiver. Then any thick subcategory in* $\mathrm{mod}\, kQ$ *is exact abelian.*

Proof: Since any indecomposable module in $\mathrm{mod}\, kQ$ is preprojective, the claim follows. □

3.2 Thick subcategories for Euclidean quivers

As we have seen, thick hereditary categories generated by preprojective or preinjective modules are exact abelian. But in general not all modules of finite dimensional algebras are preinjective or preprojective, as we see from the following proposition.

Proposition 3.2.1 *[AS, Chapter VIII.2, Corollary 2.10] Let A be a representation infinite algebra. Then there exists an infinite family of pairwise non-isomorphic indecomposable A-modules that are neither preprojective nor preinjective.*

Therefore, it may happen that the exact abelian subcategories in $\operatorname{mod} A$ are not generated only by preprojective or preinjective modules.

Definition 3.2.2 Let A be an arbitrary (not necessarily hereditary) k-algebra. A connected component $\mathcal{C}$ of $\Gamma(\operatorname{mod} A)$ is called **regular component**, if $\mathcal{C}$ contains neither projective nor injective modules. An indecomposable A-module is called **regular indecomposable**, if it belongs to a regular component of $\Gamma(\operatorname{mod} A)$ and an arbitrary A-module is called **regular**, if it is a direct sum of indecomposable A-modules. A non-zero regular module having no proper regular submodules is said to be **regular simple**.

For any regular module X, there exists a chain

$$X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_{\ell-1} = \supsetneq X_\ell = 0$$

of regular submodules of X such that X_{i-1}/X_i is simple regular for any i with $1 \geq i \geq \ell$, and ℓ is called the **regular length** of X, which we denote by $r\ell(X)$.

Let A be a representation-infinite hereditary algebra. We denote by $\mathcal{R}(A)$ the family of all the regular components of $\Gamma(\operatorname{mod} A)$ and by $\operatorname{add}(\mathcal{R}(A))$ the full subcategory of $\operatorname{mod} A$ whose objects are all the regular A-modules.

The following proposition tells us more about Hom-spaces between different components in $\Gamma(\operatorname{mod} A)$.

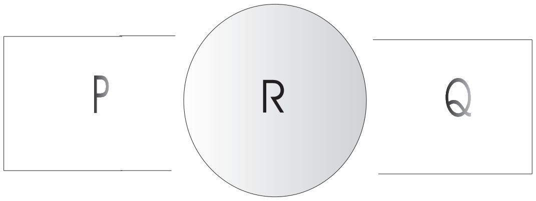
Proposition 3.2.3 *[AS, Chapter VIII.2, Corollary 2.13] Let A be as before and L, M and N be three indecomposable A-modules.*

(a) If L is preprojective and M is regular, then $\operatorname{Hom}_A(M, L) = 0$.

(b) If L is preprojective and N is preinjective, then $\operatorname{Hom}_A(N, L) = 0$.

(c) If M is regular and N is preinjective, then $\operatorname{Hom}_A(N, M) = 0$.

The picture visualises the shape of the Auslander-Reiten quiver of $\operatorname{mod} A$.



The proposition above is more briefly expressed by writing:

$\mathrm{Hom}_A(\mathcal{R}(A),\mathcal{P}(A)) = 0$, $\mathrm{Hom}_A(\mathcal{Q}(A),\mathcal{P}(A)) = 0$, $\mathrm{Hom}_A(\mathcal{Q}(A),\mathcal{R}(A)) = 0$. Using the Auslander-Reiten formula and the previous proposition, we get immediately:

$$\mathrm{Ext}^1_A(\mathcal{P}(A),\mathcal{R}(A)) = 0, \mathrm{Ext}^1_A(\mathcal{Q}(A),\mathcal{R}(A)) = 0, \mathrm{Ext}^1_A(\mathcal{P}(A),\mathcal{Q}(A)) = 0.$$

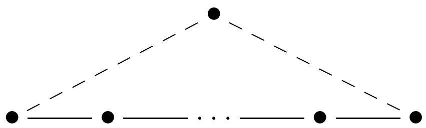
The behaviour of the Auslander-Reiten translate τ on the regular component is recorded in the following proposition, see [AS, Chapter VIII.2, Corollary 2.14].

Proposition 3.2.4 *Let A be representation-infinite hereditary algebra. The the Auslander-Reiten translations τ and τ^{-1}, induce mutually inverse equivalences of categories*

$$\begin{tikzcd} \mathrm{add}(\mathcal{R}(A)) \arrow[r, shift left, "\tau"] & \mathrm{add}(\mathcal{R}(A)) \arrow[l, shift left, "\tau^{-1}"] \end{tikzcd} .$$

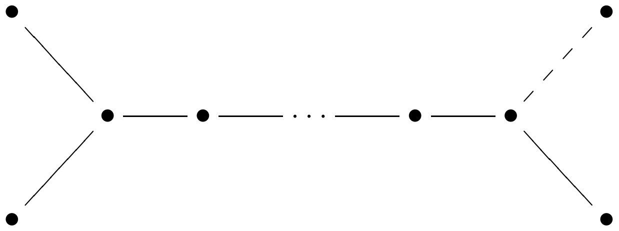
There are few cases of infinite-dimensional algebras in which the regular component is well-known. Examples of such algebras are **tame hereditary** algebras, which are the path algebras of the quivers, whose underling graph are Euclidian diagrams (one point extensions of Dynkin diagrams, see A.2). We list the Euclidian quivers, the dotted lines shows how these diagrams are obtained from the Dynkin diagrams.

$\tilde{\mathbb{A}}_n$:



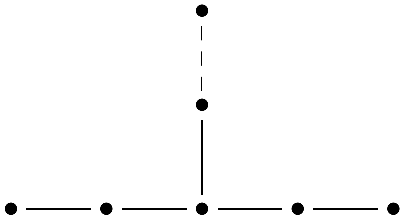
$(n \geq 1)$

$\tilde{\mathbb{D}}_n$:

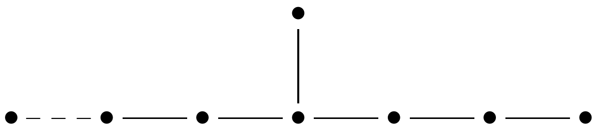


$(n \geq 4)$

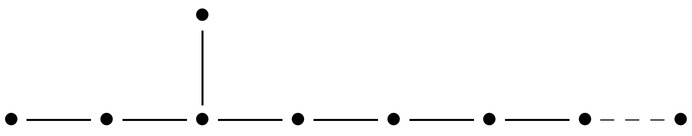
$\tilde{\mathbb{E}}_6$:



$\tilde{\mathbb{E}}_7$:



$\tilde{\mathbb{E}}_8$:



The index refers to the number of points minus one (thus $\tilde{\mathbb{A}}_n$ has $n+1$ points).

In the next theorem, we collect the basic properties of the module category over the path algebra of Euclidian type. Before that we need the following two definitions.

Definition 3.2.5 Two components $\mathcal{C}$ and $\mathcal{C}'$ of the Auslander-Reiten quiver of an algebra A is said to be **orthogonal** if $\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}') = 0$ and $\mathrm{Hom}_A(\mathcal{C}', \mathcal{C}) = 0$, that is, $\mathrm{Hom}_A(C, C') = 0$ and $\mathrm{Hom}_A(C', C) = 0$, for any module $C \in \mathcal{C}$ and any module $C' \in \mathcal{C}'$.

Definition 3.2.6 Let $\mathcal{T} = \{\mathcal{T}_i\}_{i\in\Lambda}$ be a family of stable tubes and $(m_1, \ldots, m_s)$ a sequence of integers with $1 \leq m_1 \leq \cdots \leq m_s$. We say that $\mathcal{T}$ is of **tubular type** $(m_1, \ldots, m_s)$ if $\mathcal{T}$ admits s tubes $\mathcal{T}_{i_1}, \ldots, \mathcal{T}_{i_s}$ of ranks $m_1, \ldots, m_s$, respectively, and the remaining tubes $\mathcal{T}_i$ of $\mathcal{T}$, with $i \notin \{i_1, \ldots, i_s\}$, are **homogeneous**, that is, of rank 1.

Theorem 3.2.7 *[SS, Chapter XII.3] Let Q be an acyclic quiver whose underlying graph $\overline{Q}$ is Euclidean, and $A = kQ$ be the path algebra of Q.*

(a) The Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, A)$ *of A consists of the following three types of components:*

- *a preprojective component $\mathcal{P}(A)$ containing all indecomposable projective modules,*
- *a preinjective component $\mathcal{Q}(A)$ containing all indecomposable injective modules, and*
- *a unique $\mathbb{P}_1(k)$-family*

  $$\mathcal{T}^Q = \{\mathcal{T}^Q_\lambda\}_{\lambda\in\mathbb{P}_1(k)}$$

  of pairwise orthogonal tubes, in the regular part $\mathcal{R}(A)$ of $\Gamma(\mathrm{mod}\, A)$.

(b) The tubes are exact abelian extension closed subcategories of $\mathrm{mod}\, A$. *Any indecomposable regular module is uniserial.*

(c) Let $m_Q = (m_1, \ldots, m_s)$ be the tubular type of the $\mathbb{P}_1(k)$-family $\mathcal{T}^Q$. Then

- $m_Q = (p, q)$ *if* $\overline{Q} = \tilde{\mathbb{A}}_m$, $m \geq 1$, $p = \min\{p', p''\}$, *and* $q = \max\{p', p''\}$, *where p' and p'' are the numbers of counterclockwise-oriented arrows in Q and clockwise-oriented arrows in Q, respectively,*
- $m_Q = (2, 2, m-2)$, *if* $\overline{Q} = \tilde{\mathbb{D}}_m$ *and* $m \geq 4$,
- $m_Q = (2, 3, 3)$, *if* $\overline{Q} = \tilde{\mathbb{E}}_6$,
- $m_Q = (2, 3, 4)$, *if* $\overline{Q} = \tilde{\mathbb{E}}_7$, *and*

- $m_Q = (2,3,5)$, *if* $\overline{Q} = \tilde{\mathbb{E}}_8$.

In other words, in Euclidean quiver case, kernels and cokernels of morphisms between regular modules are again regular and there are neither homomorphisms nor extensions between different tubes. The number of non-homogeneous tubes is finite.

From now on, we assume that Q is an Euclidean quiver. We adopt some notations from the previous chapter. We let $\mathcal{T}_r$ be the tube of rank r in the regular component of $\operatorname{mod} kQ$. We denote by $\{T_1, T_2, \ldots, T_r\}$ the set of simples of $\mathcal{T}_r$, and assume that $\tau(T_i) = T_{i-1}$ for any $1 \leq i \leq r$, where as before indices are taken modulo r. Since the tubes are uniserial categories, any indecomposable object in $\mathcal{T}_r$ is uniquely determined by its socle and length. As in the first chapter, $T_i[\ell]$ denotes the indecomposable object with socle T_i and length ℓ. Recall that the regular simple composition factors of a regular module is called the **regular support**. For example, $T_i[\ell]$ has support $\{T_i, T_{i+1}, \ldots, T_{i+\ell-1}\}$.

In this section, we aim to prove that any thick subcategory of $\operatorname{mod} kQ$ is exact abelian. First, we restate theorem 2.2.10 and lemma 2.3.1 from the previous chapter.

Proposition 3.2.8 *Let $\mathcal{T}_r$ be a tube of rank r in* $\operatorname{mod} kQ$*. Then any thick subcategory of $\mathcal{T}_r$ is exact abelian in $\mathcal{T}_r$ and hence in* $\operatorname{mod} kQ$*. More precisely, for any connected thick subcategory $\mathcal{C}$ of $\mathcal{T}_r$,*

(1) there exists a sequence $\{T_{i_1}[\ell_1], \ldots, T_{i_s}[\ell_s]\} \subseteq \mathcal{T}_r$ of indecomposable objects with $i_k + \ell_k = i_{k+1}$ for any k and $\ell_1 + \ell_2 + \cdots + \ell_s \leq r$;

(2) $\mathcal{C}$ is either equivalent to $\operatorname{mod} k\mathbb{A}_s$ *for the Dynkin quiver of directed $\mathbb{A}_s$ type, or to a tube of rank s; moreover, $\mathcal{C}$ is equivalent to a tube if and only if $\ell_1 + \cdots + \ell_s = r$.*

Before proving the next proposition, we recall the following fact. Let R be an indecomposable module in $\operatorname{mod} kQ$ and let $\mathcal{T}_r \subseteq \mathcal{R}$ be the unique tube of rank r that contains R. If $r\ell(R) < r$, then R is exceptional.

Proposition 3.2.9 *Let $S \subseteq \mathcal{P}$ be an arbitrary set and $E = (X_1, \ldots, X_k) \subseteq \mathcal{T}_r$ an exceptional sequence with pairwise disjoint regular supports. Then there exists an exceptional sequence $E' \subseteq \mathcal{P} \cup \mathcal{T}_r$ such that* $\operatorname{Thick}(S, E) = \operatorname{Thick}(E')$*.*

Proof: Since X_i's have pairwise disjoint regular supports, we see that E is orthogonal, that is, $\operatorname{Hom}_{kQ}(X_i, X_j) = 0$ for any $1 \leq i \neq j \leq k$. To prove the proposition, we use the induction on the sum of the lengths of X_i's.

If $\operatorname{Ext}^1_{kQ}(X_i, P) = 0$ for any $P \in S$ and $X_i \in E$, then by applying proposition 3.1.10, we have an exceptional sequence $F \subseteq \mathcal{P}$ such that $\operatorname{Thick}(F) = \operatorname{Thick}(S)$. Since $S \subseteq E^{\perp}$, then $\operatorname{Thick}(S) \subseteq E^{\perp}$, and hence $E' = (F, E)$ forms an exceptional sequence and $\operatorname{Thick}(E') = \operatorname{Thick}(S, E)$.

Now, assume that there exists some $P \in S$ and $X_i \in E$ such that $\mathrm{Ext}^1_{kQ}(X_i, P) \neq 0$. Taking a non-split short exact sequence $0 \to P \to P_1 \oplus R \to X_i \to 0$ with P_1 preprojective and R regular (the middle term can be written in this way since it has no preinjective direct summands). Notice that $\mathrm{Ker}(R \to X_i) \subseteq P$ and since $\mathrm{Ker}(R \to X_i)$ is again regular, it follows that $\mathrm{Ker}(R \to X_i) = 0$. Moreover since the sequence is non-split, R is a proper submodule of X_i.

We set S_1 to be the union of S and the indecomposable direct summands of P_1 and $E_1 = (X_1, \dots, X_{i-1}, R, X_{i+1}, \dots, X_k)$ if $R \neq 0$, or to be the union of S and the indecomposable direct summands of P_1 and $E_1 = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_k)$ if $R = 0$. It follows that $\mathrm{Thick}(S, E) = \mathrm{Thick}(S_1, E_1)$, and again E_1 is an orthogonal sequence with pairwise disjoint regular supports. In both cases the total sum of lengths of elements of E_1 is strictly less than the one of E, since in case $R \neq 0$, by construction R is a proper submodule of X_i. Repeat the argument, we get to the case such that $S \subseteq E^{\perp}$ after finite step, and the conclusion follows. Now, by the inductional hypothesis, there is an exceptional sequence $E' \subseteq \mathcal{P} \cup \mathcal{T}_r$, such that $\mathrm{Thick}(E') = \mathrm{Thick}(S, E) = \mathrm{Thick}(S_1, E_1)$. The proof follows. □

As a consequence, we get the following corollary.

Corollary 3.2.10 *Let $P \in \mathcal{P}$ and $R \in \mathcal{R}$ such that $\mathrm{Ext}^1_{kQ}(R, P) \neq 0$. Then there exists an exceptional sequence E such that $\mathrm{Thick}(P, R) = \mathrm{Thick}(E)$.*

Proof: Let $\mathcal{T}_r$ be the unique tube with rank r, which contains R. Assume that the simple objects $T_1, T_2, \dots, T_r$ of $\mathcal{T}_r$ are ordered in such a way that $\tau T_i = T_{i-1}$. Without loss of generality, we assume that $R = T_1[\ell]$ for some ℓ.

There are three cases.

Case 1. If $1 < \ell < r$, then R is an exceptional module, and hence there exists an exceptional sequence E such that $\mathrm{Thick}(P, R) = \mathrm{Thick}(E)$ by proposition 3.2.9.

Case 2. $\ell = mr$, for $m \geq 1$. Then $\mathrm{Thick}(T_1[mr]) = \mathrm{Thick}(T_1[r])$. Without loss of generality, we may assume that $\ell = r$. By assumption, $\mathrm{Ext}^1_{kQ}(R, P) \neq 0$ and we may take a non-split short exact sequence $0 \to P \to P_1 \oplus R_1 \to R \to 0$. With the same argument as in the proof of proposition 3.2.9, we can show that $\mathrm{Thick}(P, R) = \mathrm{Thick}(P, P_1, R_1)$ with R_1 a proper regular submodule of R. Now, R_1 has no self extensions and again using proposition 3.2.9 there exists an exceptional sequence E such that $\mathrm{Thick}(E) = \mathrm{Thick}(P, P_1, R_1) = \mathrm{Thick}(P, R)$.

Case 3. $\ell = rm + s$, for some $1 \leq s < r$. Then $\mathrm{Thick}(T_1[rm + s]) = \mathrm{Thick}(T_1[s], T_{s+1}[r - s])$. Since $T_1[\ell]$ has a filtration with factors $T_1[s]$ and $T_{s+1}[r-s]$ and $\mathrm{Ext}^1_{kQ}(T_1[\ell], P) \neq 0$, we have $\mathrm{Ext}^1_{kQ}(T_1[s], P) \neq 0$ or $\mathrm{Ext}^1_{kQ}(T_{s+1}[r - s], P) \neq 0$.

First, assume that $\mathrm{Ext}^1_{kQ}(T_1[s], P) \neq 0$. We take a non-split short exact sequence $0 \to P \to P_1 \oplus R_1 \to T_1[s] \to 0$. By the same argument as before, we get that R_1 is a proper regular submodule of $T_1[s]$, and hence $\{R_1, T_{s+1}[r - s]\}$ forms an exceptional sequence. Now, by applying proposition 3.2.9, we get that there exists an exceptional sequence E such that $\operatorname{Thick}(E) = \operatorname{Thick}(P, P_1, R_1) = \operatorname{Thick}(P, R)$.

The same argument works for the case that $\operatorname{Ext}^1_{kQ}(T_{s+1}[r-s], P) \neq 0$, which completes the proof. □

As in the previous chapter, we shall frequently use the Happel-Ringel's lemma, so we recall it here.

Lemma 3.2.11 *(Happel-Ringel) Let $\mathcal{H}$ be a hereditary abelian category. Assume that X, Y are indecomposable objects in $\mathcal{H}$ and $\operatorname{Ext}^1_{\mathcal{H}}(Y, X) = 0$. Then any non-zero morphism $f\colon X \to Y$ is either monomorphism or epimorphism.*

In Chapter 1 we discussed, that any thick subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is exact abelian if and only if $\mathcal{C}$ is closed under kernels (or equivalently closed under images, or closed under cokernels). In the next proposition we prove that any thick subcategory in $\operatorname{mod} kQ$, where Q is an Euclidian quiver is closed under kernels, and hence it is exact abelian. As we shall see in theorem 3.3.1, this statement holds true for any abelian hereditary category. But the proof, we shall present, is explicit and we shall use it in the next chapter to prove the main theorem there. We also refer the reader to A.4, where basic homological facts, which are frequently used in the proof, are collected.

Proposition 3.2.12 *Let Q be an Euclidian quiver, X and Y kQ-modules and $f\colon X \to Y$ a non-zero morphism between them. Then* $\operatorname{Ker} f \in \operatorname{Thick}(X, Y)$.

Proof: We use induction on $d = \dim(X) + \dim(Y)$, where the dimension is over k. Clearly, the assertion holds in case either X or Y is simple. Now, assume that the assertion is true for any morphism $f\colon X' \to Y'$ with $\dim(X') + \dim(Y') < d$.

First, we assume that X is decomposable and write $X = X_1 \oplus X_2$ with X_1 and X_2 non-zero. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "{(1,0)}"] \arrow[d, "f_1"] & X_1 \oplus X_2 \arrow[r, "\binom{0}{1}"] \arrow[d, "f=\binom{f_1}{f_2}"] & X_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r, "\mathrm{Id}"] & Y \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Applying the snake lemma, we get the exact sequence

$$0 \to \operatorname{Ker} f_1 \to \operatorname{Ker} f \to X_2 \to \operatorname{Coker} f_1 \to \operatorname{Coker} f \to 0.$$

From $\dim(X_1) < \dim(X)$ follows by induction that $\operatorname{Coker} f_1 \in \operatorname{Thick}(X_1, Y) \subseteq \operatorname{Thick}(X, Y)$. Now, since $\operatorname{Coker} f = \operatorname{Coker}(X_2 \to \operatorname{Coker} f_1)$ and $\dim(X_2) < \dim(X)$, $\dim(\operatorname{Coker} f_1) \leq \dim Y$, we get $\operatorname{Coker} f \in \operatorname{Thick}(X_2, \operatorname{Coker} f_1) \subseteq \operatorname{Thick}(X, Y)$, and hence $\operatorname{Ker} f \in \operatorname{Thick}(X, Y)$.

The dual version of the above argument shows that if Y is decomposable, then $\operatorname{Ker} f \in \operatorname{Thick}(X, Y)$.

Now, we may assume that both X and Y are indecomposable. If X and Y are preprojective (preinjective), then by proposition 3.1.10 (remark 3.1.11) we have $\operatorname{Ker} f \in \operatorname{Thick}(X, Y)$. If X and Y are regular, then by proposition 3.2.8 we have $\operatorname{Ker} f \in \operatorname{Thick}(X, Y)$. Having in mind proposition 3.2.3, the only cases left are:

Case 1. $X = P \in \mathcal{P}$ and $Y = R \in \mathcal{R}$.

Case 2. $X = R \in \mathcal{R}$ and $Y = Q \in \mathcal{Q}$.

Case 3. $X = P \in \mathcal{P}$ and $Y = Q \in \mathcal{Q}$.

We proceed with a case-by-case analysis.

Case 1. $P \in \mathcal{P}$, $R \in \mathcal{R}$ and $0 \neq f \colon P \to R$.

If $\operatorname{Ext}^1_{kQ}(R, P) = 0$, then by Happel-Ringel's lemma, f is either injective or surjective, so in both cases $\operatorname{Ker} f \in \operatorname{Thick}(R, P)$.

Now, we assume that $\operatorname{Ext}^1_{kQ}(R, P) \neq 0$. Applying corollary 3.2.10, we show that $\operatorname{Thick}(P, R)$ is exact abelian, and hence $\operatorname{Ker} f \in \operatorname{Thick}(R, P)$.

Case 2. Dual to Case 1.

Case 3. $P \in \mathcal{P}$, $Q \in \mathcal{Q}$ and $0 \neq f \colon P \to Q$.

Again by Happel-Ringel's lemma, we may assume that $\operatorname{Ext}^1_{kQ}(Q, P) \neq 0$, so let η be a non-split short exact sequence $\eta : 0 \to P \to M \to Q \to 0$. There are two possibilities: **(i)** M is indecomposable or **(ii)** M is decomposable, and we deal with these cases separately.

(i) M is indecomposable.

By proposition 3.1.10 and remark 3.1.11, if M is either preprojective or preinjective, then $\operatorname{Thick}(P, Q)$ is exact abelian and therefore $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$.

Now, assume that M is regular. If M has no self-extensions, we can apply proposition 3.2.9 to get that $\operatorname{Thick}(P, Q) = \operatorname{Thick}(P, M)$ is exact abelian, and hence $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$. If M has self-extensions, by applying the functor $\operatorname{Hom}_{kQ}(M, -)$ on $0 \to P \to M \to Q \to 0$, we get an exact sequence $\operatorname{Ext}^1_{kQ}(M, P) \to \operatorname{Ext}^1_{kQ}(M, M) \to 0$, which forces that $\operatorname{Ext}^1_{kQ}(M, P) \neq 0$. By corollary 3.2.10, we have that $\operatorname{Thick}(P, M)$ is exact abelian and hence $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$.

(ii) M is decomposable.

Suppose $M = M_1 \oplus M_2$ for some $M_1, M_2 \neq 0$. The proof that $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$ is divided into 3 steps.

Step 1. Let $0 \to P \overset{\binom{g_1}{g_2}}{\to} M_1 \oplus M_2 \overset{(h_1, h_2)}{\to} Q \to 0$ be a non-split short exact sequence. If one of $\operatorname{Ker} g_1, \operatorname{Ker} g_2, \operatorname{Coker} h_1, \operatorname{Coker} h_2$ is non-zero and contained in $\operatorname{Thick}(P, Q)$, then so is $\operatorname{Ker} f$.

First, assume that $0 \neq \operatorname{Ker} g_1 \in \operatorname{Thick}(P, Q)$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} g_1 \arrow[r, "i"] \arrow[d, "f \circ i"] & P \arrow[r, "\pi"] \arrow[d, "f"] & U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q \arrow[r, "\mathrm{Id}"] & Q \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma, we get a long exact sequence

$$0 \to \operatorname{Ker}(f \circ i) \to \operatorname{Ker} f \to U \to \operatorname{Coker}(f \circ i) \to \operatorname{Coker} f \to 0.$$

Since Q is indecomposable, we claim that $\operatorname{Ker} g_1$ is a proper submodule of P. Otherwise, if $\operatorname{Ker} g_1 = P$, then $\operatorname{Im} h_1 \cong M_1$ and $\operatorname{Im} h_2 \cap \operatorname{Im} h_1 = 0$, and hence $Q \cong \operatorname{Im} h_1 \oplus \operatorname{Im} h_2$. Since Q is indecomposable, we have that $\operatorname{Im} h_2 = 0$, $M_1 \cong Q$, $P \cong M_2$ and the short exact sequence splits. This leads to a contradiction. Hence we have $\dim(\operatorname{Ker} g_1) < \dim(P)$ and by induction hypothesis on the dimensions, $\operatorname{Ker}(f \circ i) \in \operatorname{Thick}(\operatorname{Ker} g_1, Q) \subseteq \operatorname{Thick}(P, Q)$ and $\operatorname{Coker}(f \circ i) \in \operatorname{Thick}(P, Q)$. Moreover $\operatorname{Ker} g_1 \neq 0$ implies that $\dim(U) < \dim(P)$, together with the facts that $\dim(\operatorname{Coker}(f \circ i)) \leq \dim(Q)$ and $\operatorname{Coker} f = \operatorname{Coker}(U \to \operatorname{Coker}(f \circ i))$, it follows from that $\operatorname{Coker} f \in \operatorname{Thick}(U, \operatorname{Coker}(f \circ i)) \subseteq \operatorname{Thick}(P, Q)$.

A dual version works in case that $0 \neq \operatorname{Coker} h_1 \in \operatorname{Thick}(P, Q)$ by using the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P \arrow[r, "\mathrm{Id}"] \arrow[d, "f"] & P \arrow[r] \arrow[d, "\pi \circ f"] & 0 \\
0 \arrow[r] & V \arrow[r, "i"] & Q \arrow[r, "\pi"] & \operatorname{Coker} h_1 \arrow[r] & 0
\end{tikzcd}$$

and the snake lemma. The other cases are treated the same.

Step 2. Let $0 \to P \overset{(g_1, g_2)}{\longrightarrow} M_1 \oplus M_2 \overset{\binom{h_1}{h_2}}{\to} Q \to 0$ be a non-split short exact sequence. If $\min\{\dim M_1, \dim M_2\} < \max\{\dim P, \dim Q\}$, then $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$.

First assume that $\dim M_1 < \dim P$. By the induction hypothesis on the dimension, follows that $\operatorname{Ker} h_1 \in \operatorname{Thick}(M_1, Q) \subseteq \operatorname{Thick}(P, Q)$. Therefore, if $\operatorname{Coker} h_1 \neq 0$, then $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$ by Step 1. Now, assume that $\operatorname{Coker} h_1 = 0$.

By the property of push-out and pull-back, we know that $\operatorname{Ker} g_1 = 0$ if and only if $\operatorname{Ker} h_2 = 0$ and $\operatorname{Coker} g_1 = 0$ if and only if $\operatorname{Coker} h_1 = 0$. Now, $\operatorname{Coker} h_1 = 0$ implies that $\operatorname{Coker} g_2 = 0$ and hence $\operatorname{Ker} g_2 \in \operatorname{Thick}(P, M_2) \subseteq \operatorname{Thick}(P, Q)$. By Step 1, to show that $\operatorname{Ker} f \subseteq \operatorname{Thick}(P, Q)$, it suffices to show that $\operatorname{Ker} g_2 \neq 0$. In fact, if $\operatorname{Ker} g_2 = 0$, then g_2 is an isomorphism and hence the short exact sequence splits, which gives a contradiction and the assertion follows.

A dual version of the above argument works for the case $\dim M_1 < \dim Q$.

Step 3. Let $0 \to P \overset{(g_1, g_2)}{\longrightarrow} M_1 \oplus M_2 \overset{\binom{h_1}{h_2}}{\to} Q \to 0$ be a non-split short exact sequence with $\dim(P) = \dim(M_1) = \dim(M_2) = \dim(Q)$. We claim that $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$.

Applying Step 2 we may assume that M_1 and M_2 are both indecomposable. If both M_1 and M_2 are preinjective, then $\operatorname{Thick}(P, Q) = \operatorname{Thick}(M_1, M_2, Q)$ and hence is exact abelian by remark 3.1.11, which implies that $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$. Otherwise if one of them, say M_1, is preprojective or regular, then by Case 1, we know that $\operatorname{Ker} g_1 \in \operatorname{Thick}(P, Q)$. We claim that $\operatorname{Ker} g_1 \neq 0$. Otherwise the assumption $\dim(P) = \dim(M_1)$ implies that g_1 is an isomorphism and hence the short exact sequence splits. It follows that $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$ by Step 1.

So far, we have shown that if M is not indecomposable, we are either in the situation of Step 2 or Step 3, and in both cases $\operatorname{Ker} f \in \operatorname{Thick}(P, Q)$.

Now, we have shown that $\operatorname{Ker} f \in \operatorname{Thick}(X, Y)$ holds for any $f \colon X \to Y$ with $\dim(X) + \dim(Y) = d$, which finishes the proof. □

As we already discussed, any thick category closed under arbitrary kernels is exact abelian. The previous proposition gives us immediately the following result.

Corollary 3.2.13 *Let $\mathcal{C}$ be a thick subcategory in $\operatorname{mod} kQ$. Then $\mathcal{C}$ is exact abelian.*

Let us summarize the results obtained so far.

Theorem 3.2.14 *Let k be an algebraically closed filed, Q a finite quiver and kQ its path algebra.*

(i) Let S be a set of kQ-modules with $S \subseteq \mathcal{P}$ or $S \subseteq \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ denote the preprojective and preinjective component of $\operatorname{mod} kQ$ respectively. Then $\operatorname{Thick}(S)$ is exact abelian.

(ii) If Q is either Dynkin or Euclidean quiver, then any thick subcategory of $\operatorname{mod} kQ$ is exact abelian.

3.2.1 Classification of thick subcategories

A result by Ingalls and Thomas says that for an Euclidean quiver Q, there exists a one-to-one correspondence between the non-crossing partitions associated to Q and the “finitely generated wide subcategories” [IT, Theorem 1.1] of $\operatorname{mod} kQ$. Note that the “wide subcategories” refer to the exact abelian extension closed subcategories in our sense, and “finitely generated” means that the subcategory has a projective generator. As we shall prove, any thick subcategory of kQ has either a projective generator, or consists of regular modules.

Theorem 3.2.15 *Let k be an algebraically closed field and Q an Euclidean quiver. Let $\mathcal{C}$ be a thick subcategory of $\operatorname{mod} kQ$. Then at least one of the following holds:*

(i) There exists an exceptional sequence E, such that $\mathcal{C} = \operatorname{Thick}(E)$.

(ii) Any object in $\mathcal{C}$ is regular.

Proof: In the previous section, we showed that any thick subcategory of $\mathrm{mod}\, kQ$ is exact abelian. In particular, $\mathcal{C}$ is exact abelian, and we can consider its simple objects. Let $S_{\mathcal{P}}$ be a complete set of simples in $\mathcal{C}$, which are preprojective in $\mathrm{mod}\, kQ$. By using the order we defined on $\mathcal{P}$, we know that $S_{\mathcal{P}}$ forms an exceptional sequence. We denote by $S_{\mathcal{R}}$ and $S_{\mathcal{Q}}$ the set of simples which are regular and preinjective respectively.

We claim that if $S_{\mathcal{P}} \cup S_{\mathcal{Q}} \neq \emptyset$, then we can make $S_{\mathcal{Q}} \cup S_{\mathcal{R}} \cup S_{\mathcal{P}}$ into an exceptional sequence.

Without loss of generality, we may assume that $S_{\mathcal{P}} \neq \emptyset$ and $P = \tau^{-l} P_j \in S_{\mathcal{P}}$, where P_j is a projective module and $l \geq 0$. First we show that in this case, $S_{\mathcal{R}}$ is finite. Note that in Euclidean case, $\dim_k(\mathrm{Hom}_{kQ}(P_j, R)) - \dim_k(\mathrm{Ext}^1_{kQ}(P_j, R)) > 0$ for any module R which appears in some homogeneous tube, see [CB2, Lemma 7.2], since the dimension vector of any such regular module is a multiple of δ, the minimal imaginary root associated to Q. But $\mathrm{Ext}^1_{kQ}(P_j, R) = 0$, so we conclude that $\mathrm{Hom}_{kQ}(P_j, R) \neq 0$. Since $S_{\mathcal{P}}$ and $S_{\mathcal{R}}$ are both sets of simples in the category $\mathcal{C}$, then there are no non-zero morphisms between different elements in these sets. Hence the elements of $S_{\mathcal{R}}$ are from the non-homogeneous tubes. From theorem 3.2.7, we know that there are finitely many non-homogeneous tubes in Euclidean quiver case, and since the number of elements in $S_{\mathcal{R}}$ from one tube is not greater than the rank of the tube, we conclude that $S_{\mathcal{R}}$ is finite.

Next, assume that $E = \{X_1, X_2, \ldots, X_t\} = S_{\mathcal{R}} \cap \mathcal{T}_r$ for some non-homogeneous tube $\mathcal{T}_r$ of rank r. We show that E forms an exceptional sequence after some reordering. Let $\{T_1, T_2, \ldots, T_r\}$ be the complete set of regular simple modules in $\mathcal{T}_r$, and again assume that $\tau T_i = T_{i-1}$. The indices are taken modulo r and we identify $T_0 = T_r$.

Since the dimension vector of $T = \oplus_{i=1}^{r} T_i$ equals δ, see [CB2, Lemma 9.3], again by [CB2, Lemma 7.2], we have that $\mathrm{Hom}_{kQ}(P_j, T) \neq 0$. Therefore there exists some T_i such that $\mathrm{Hom}_{kQ}(P_j, T_i) \neq 0$. Moreover, for any object X in $\mathcal{T}_r$ which has T_i as a composition factor, $\mathrm{Hom}_{kQ}(P_j, X) \neq 0$. This is equivalent to say that $\mathrm{Hom}_{kQ}(\tau^{-l} P_j, X) \neq 0$ for any object in $\mathcal{T}_r$ which has T_{i+l} as a composition factor. Since P and all X_i's are simples in $\mathcal{C}$, we have $\mathrm{Hom}_{kQ}(P, X_i) = 0$ for any $1 \leq i \leq t$, which forces that the regular support of $\{X_1, X_2, \ldots, X_t\}$ to be contained in $\{T_1, \ldots, T_r\} \setminus \{T_{i+l-1}\}$. Now, it is not difficult to show that E is an exceptional sequence, since the subcategory $\mathrm{Thick}(\{T_1, \ldots, T_r\} \setminus \{T_{i+l-1}\})$ is equivalent to the module category of the quiver of directed $\mathbb{A}_{r-1}$ type.

Since for each non-homogeneous tube $\mathcal{T}_r$, we showed that $S_{\mathcal{R}} \cap \mathcal{T}_r$ forms an exceptional sequence, it follows that $S_{\mathcal{R}}$ forms an exceptional sequence since there exists no extensions between different tubes. Combined with the fact $\mathrm{Ext}^1_{kQ}(\mathcal{P}, \mathcal{Q}) = \mathrm{Ext}^1_{kQ}(\mathcal{P}, \mathcal{R}) = \mathrm{Ext}^1_{kQ}(\mathcal{R}, \mathcal{Q}) = 0$, we have that $S_{\mathcal{Q}} \cup S_{\mathcal{R}} \cup S_{\mathcal{P}}$ forms an exceptional sequence.

Now, we assume that both $S_{\mathcal{P}}$ and $S_{\mathcal{Q}}$ are empty, and in this case, any objects X in $\mathcal{C}$ has a filtration with factors in $S_{\mathcal{R}}$ and hence X is regular. We are done. □

Remark 3.2.16 (1) Notice that there are thick subcategories of $\operatorname{mod} kQ$, which consist of regular modules and are generated by exceptional sequences. All these subcategories are given by direct sums of bounded thick subcategories of non-homogeneous tubes, which we classified in the previous chapter.

(2) The thick subcategories generated by exceptional sequences coincide with the so called "finitely generated wide subcategories", as defined in [IT]. In fact, a thick subcategory generated by an exceptional sequence is isomorphic to the module category of some quiver, and hence has a projective generator. Conversely, if a thick subcategory $\mathcal{C}$ is not generated by any exceptional sequence, then by the last theorem and proposition 3.2.8, $\mathcal{C}$ has a tube as a direct summand, and clearly a tube has no finite projective generator. We comment that all these categories refer to unbounded thick subcategories, which we classified in the previous chapter.

Now, having in mind these remarks, the result of Colin Ingalls and Hugh Thomas [IT, Theorem 1.1] and theorem 2.2.13, we also obtain the combinatorial classification of thick subcategories in $\operatorname{mod} kQ$.

Corollary 3.2.17 *Let k be an algebraically closed field, Q an Euclidian quiver and $\mathcal{C}$ a connected thick subcategory in* $\operatorname{mod} kQ$.

(i) If $\mathcal{C}$ has a projective generator, then $\mathcal{C}$ corresponds to a non-crossing partition of type Q.

(ii) If $\mathcal{C}$ has no projective generator, then $\mathcal{C}$ corresponds to a configuration of non-crossing arcs covering the circle.

3.3 Thick subcategories are exact abelian

We finish this chapter with pointing out a very elegant proof due to Dieter Vossieck, that any thick subcategory $\mathcal{C}$ of an abelian hereditary category $\mathcal{H}$ is exact abelian.

Theorem 3.3.1 *Let $\mathcal{H}$ be a hereditary abelian category and $\mathcal{C} \subseteq \mathcal{H}$ be a thick subcategory. Then $\mathcal{C}$ is exact abelian.*

Proof: Let X, Y be arbitrary objects in $\mathcal{C}$ and f be a non-zero morphism:

$$\begin{tikzcd}
 & X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dr] & \\
\operatorname{Ker} f \arrow[ur, hook] & & \operatorname{Im} f \arrow[ur, hook, "i"] & & \operatorname{Coker} f.
\end{tikzcd}$$

Consider the following short exact sequences:

$$\psi : 0 \to \operatorname{Ker} f \to X \overset{\pi}{\twoheadrightarrow} \operatorname{Im} f \to 0$$

$$\xi : 0 \to \operatorname{Im} f \overset{i}{\hookrightarrow} Y \to \operatorname{Coker} f \to 0.$$

Now, apply the functor $\operatorname{Hom}_{\mathcal{H}}(\operatorname{Coker} f, -)$ to ψ. Since $\mathcal{H}$ is hereditary, then the long exact sequence terminates at Ext^2-terms:

$$\cdots \to \operatorname{Ext}^1_{\mathcal{H}}(\operatorname{Coker} f, X) \overset{\operatorname{Ext}^1_{\mathcal{H}}(\operatorname{Coker} f, \pi)}{\longrightarrow} \operatorname{Ext}^1_{\mathcal{H}}(\operatorname{Coker} f, \operatorname{Im} f) \to 0.$$

We get that $\operatorname{Ext}^1_{\mathcal{H}}(\operatorname{Coker} f, \pi)$ is surjective, and hence there is $\eta \in \operatorname{Ext}^1_{\mathcal{H}}(\operatorname{Coker} f, X)$ such that $\operatorname{Ext}^1_{\mathcal{H}}(\operatorname{Coker} f, \pi)(\eta) = \xi$:

$$\begin{tikzcd}
\eta : \arrow[d, "\operatorname{Ext}^1_{\mathcal{H}}(\operatorname{Coker} f{,}\pi)"'] & 0 \arrow[r] & X \arrow[r, hook] \arrow[d, "\pi"'] & E \arrow[r] \arrow[d, dotted] & \operatorname{Coker} f \arrow[r] \arrow[d, "id"] & 0 \\
\xi : & 0 \arrow[r] & \operatorname{Im} f \arrow[r, hook, "i"] & Y \arrow[r] & \operatorname{Coker} f \arrow[r] & 0
\end{tikzcd}$$

But then Y is the push-out of $\operatorname{Im} f \overset{\pi}{\twoheadleftarrow} X \hookrightarrow E$ (see A.4) and therefore the sequence $0 \to X \to \operatorname{Im} f \oplus E \to Y \to 0$ is short exact, see [AS, A.5., Proposition 5.2]. Now, since $\mathcal{C}$ is closed under extensions and direct summands, we have that $\operatorname{Im} f$ is in $\mathcal{C}$. As we already discussed, if $\mathcal{C}$ is closed under arbitrary images, then it is automatically closed under arbitrary kernels and cokernels. The proof follows. □

Chapter 4

Exact abelian extension closed subcategories for tilted algebras

Tilting theory is one of the main tools in the representation theory of finite dimensional algebras. The main idea of the tilting theory is that when the representation theory of an algebra A is difficult to study directly, it may be convenient to replace A with another simpler algebra B and to reduce the problem on A to a problem on B. It is possible to construct a module T_A, called a tilting module, which can be thought of as being close to the Morita progenerator such that, if $B = \operatorname{End}_A(T_A)$, then the categories $\operatorname{mod} A$ and $\operatorname{mod} B$ are reasonably close to each other and there is a natural way to pass from one category to the other.

In this chapter we study exact abelian extension closed categories for tilted algebras. We show that there is a bijection between the exact abelian extension and torsion closed subcategories of $\operatorname{mod} A$, where A is a hereditary algebra and the exact abelian extension closed subcategories of the module category of its tilted algebra $B = \operatorname{End}_A(T_A)$.

4.1 Torsion pairs, tilting modules and tilted algebras

In this section, we collect same facts from tilting theory, which we shall use later. The reference for all facts is [AS, Chapter VI].

Definition 4.1.1 A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} A$ is called a **torsion pair** if the following conditions are satisfied:

(a) $\operatorname{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(b) $\operatorname{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.

(c) $\mathrm{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

Definition 4.1.2 A subfunctor t of the identity functor in $\mathrm{mod}\, A$ is called an **idempotent radical** if, for every module M_A, $t(tM) = tM$ and $t(M/tM) = 0$.

We recall that a subfunctor of the identity functor on $\mathrm{mod}\, A$ is a functor $t : \mathrm{mod}\, A \to \mathrm{mod}\, A$ that assigns to each module M a submodule $tM \subseteq M$ such that each homomorphism $M \to N$ restricts to a homomorphism $tM \to tN$. The following proposition gives us characterisation of torsion and torsion-free classes.

Proposition 4.1.3 *(a) Let $\mathcal{T}$ be a full subcategory of* $\mathrm{mod}\, A$. *The following conditions are equivalent:*

(i) $\mathcal{T}$ is a torsion class of some torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\, A$.

(ii) $\mathcal{T}$ is closed under images, direct sums, and extensions.

(iii) There exists an idempotent radical t such that $\mathcal{T} = \{M \mid tM = M\}$.

(b) Let $\mathcal{F}$ be a full subcategory of $\mathrm{mod}\, A$. *The following conditions are equivalent:*

(i) $\mathcal{F}$ is a torsion-free class of some torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\, A$.

(ii) $\mathcal{F}$ is closed under submodules, direct products, and extensions.

(iii) There exists an idempotent radical t such that $\mathcal{F} = \{N \mid tN = 0\}$.

The idempotent radical t attached to a given torsion pair is called the **torsion radical**. It follows from the definition that for any module M_A, we have $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. The uniqueness follows from the next proposition, which also says that any module can be written in a unique way as the extension of a torsion-free module by a torsion module.

Proposition 4.1.4 *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\mathrm{mod}\, A$ *and M be an A-module. There exists a short exact sequence*

$$0 \to tM \to M \to M/tM \to 0$$

with $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. This sequence is unique in a sense that, if $0 \to M' \to M \to M'' \to 0$ is exact with $M' \in \mathcal{T}$ and $M'' \in \mathcal{F}$, then the two sequences are isomorphic.

The short exact sequence $0 \to tM \to M \to M/tM \to 0$ is called the **canonical sequence** for M.

Corollary 4.1.5 *Every simple module is either torsion or torsion-free.*

A torsion pair $(\mathcal{T}, \mathcal{F})$ such that each indecomposable A-module lies either in $\mathcal{T}$ or in $\mathcal{F}$ is called **splitting**. Splitting torsion pairs are characterised as follows.

Proposition 4.1.6 *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair in* $\operatorname{mod} A$. *The following conditions are equivalent:*

(a) $(\mathcal{T}, \mathcal{F})$ *is splitting.*

(b) For each A-module M, the canonical sequence for M splits.

Next, we recall the definition of a tilting module.

Definition 4.1.7 Let A be an algebra. An A-module T is called a **partial tilting module** if the following two conditions are satisfied:

(T1) the projective dimension of T is at most 1.

(T2) $\operatorname{Ext}^1_A(T, T) = 0$.

A partial tilting module T is called a **tilting module**, if it also satisfies the following additional condition:

(T3) There exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T', T'' in $\operatorname{add}(T)$.

A tilting module is called **basic**, if each indecomposable direct summand occurs exactly once in its direct sum decomposition.

Let T be an arbitrary A-module. We define $\operatorname{Gen}(T)$ to be the class of all modules M in $\operatorname{mod} A$ generated by T, that is, the modules M such that there exists an integer $d \geq 0$ and an epimorphism $T^d \to M$ of A-modules. Dually, we define $\operatorname{Cogen}(T)$ to be the class of all modules N in $\operatorname{mod} A$ cogenerated by T, that is, the modules N such that there exist an integer $d \geq 0$ and a monomorphism $N \to T^d$ of A-modules.

Proposition 4.1.8 *Let T_A be a partial tilting module. The following are equivalent:*

(a) T_A is a tilting module.

(b) $\operatorname{Gen}(T) = \mathcal{T}(T) = \{M_A \mid \operatorname{Ext}^1_A(T, M) = 0\}$ *is a torsion class in* $\operatorname{mod} A$ *with corresponding torsion-free class* $\operatorname{Cogen}(\tau T) = \mathcal{F}(T) = \{M_A \mid \operatorname{Hom}_A(T, M) = 0\}$.

For a given tilting module, we introduce a new class of algebras.

Definition 4.1.9 Let A be a finite dimensional, hereditary k-algebra and T_A be a tilting module. The k-algebra $\operatorname{End}_A(T_A)$ is called a **tilted algebra**.

The following proposition tells us what is the effect of any tilting module T_A on $\operatorname{mod} B$.

Proposition 4.1.10 *Let A be an algebra. Any tilting A-module T_A induces a torsion pair $\mathcal{X}(T_A), \mathcal{Y}(T_A)$ in the category* $\operatorname{mod} B$, *where $B = \operatorname{End}_A(T_A)$ and*

$$\mathcal{X}(T_A) = \{X_B \mid \operatorname{Hom}_B(X, DT) = 0\} = \{X_B \mid X \otimes_B T = 0\},$$

$$\mathcal{Y}(T_A) = \{Y_B \mid \operatorname{Ext}^1_B(Y, DT) = 0\} = \{Y_B \mid \operatorname{Tor}^B_1(Y, T) = 0\}.$$

The next theorem, known as Brenner-Butler theorem or a tilting theorem, is a milestone in the tilting theory.

Theorem 4.1.11 *(Brenner-Butler) Let A be an algebra, T_A be a tilting module, $B =$* $\operatorname{End}_A(T_A)$, *and $(\mathcal{T}(T_A), \mathcal{F}(T_A))$, $\mathcal{X}(T_A), \mathcal{Y}(T_A)$ be induced torsion pairs in* $\operatorname{mod} A$ *and* $\operatorname{mod} B$, *respectively. Then T_A has the following properties:*

(a) ${}_BT$ is a tilting module, and the canonical k-algebra homomorphism $A \longrightarrow \operatorname{End}({}_BT)^{\mathrm{op}}$ defined by a $a \mapsto (t \mapsto ta)$ is an isomorphism.

(b) The functors $\operatorname{Hom}_A(T, -)$ and $- \otimes_B T$ induce quasi-inverse equivalences between $\mathcal{T}(T_A)$ and $\mathcal{Y}(T_A)$.

(c) The functors $\operatorname{Ext}^1_A(T, -)$ and $\operatorname{Tor}^B_1(-, T)$ induce quasi-inverse equivalences between $\mathcal{F}(T_A)$ and $\mathcal{X}(T_A)$.

The following proposition asserts that the composition of any two of the four functors $\operatorname{Hom}_A(T, -)$, $\operatorname{Ext}^1_A(T, -)$, $- \otimes_B T$ and $\operatorname{Tor}^B_1(-, T)$, which are not quasi-inverse to each other, vanishes.

Proposition 4.1.12 *(a) Let M be an arbitrary A-module. Then*

(i) $\operatorname{Tor}^B_1(\operatorname{Hom}_A(T, M), T) = 0$.

(ii) $\operatorname{Ext}^1_A(T, M) \otimes_B T = 0$.

(iii) The canonical sequence of M in $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ is

$$0 \longrightarrow \operatorname{Hom}_A(T, M) \otimes_B T \longrightarrow M \longrightarrow \operatorname{Tor}^B_1(\operatorname{Ext}^1_A(T, M), T) \longrightarrow 0.$$

(b) Let X be an arbitrary B-module. Then

(i) $\operatorname{Hom}_A(T, \operatorname{Tor}^B_1(X, T)) = 0$.

(ii) $\operatorname{Ext}^1_A(T, X \otimes_B T) = 0$.

(iii) The canonical sequence of X in $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ is

$$0 \to \operatorname{Ext}_A^1(T, \operatorname{Tor}_1^B(X, T)) \to X \to \operatorname{Hom}_A(T, X \otimes_B T) \to 0.$$

We introduce two types of tilting modules.

Definition 4.1.13 Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End}_A(T_A)$. Then

(a) T_A is said to be **separating** if the induced torsion pair $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ in $\operatorname{mod} A$ is splitting, and

(b) T_A is said to be **splitting** if the induced torsion pair $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ in $\operatorname{mod} B$ is splitting.

The next proposition tells us when a tilting module is separating or splitting.

Proposition 4.1.14 *Let A be an algebra, T_A be a tilting A-module, and $B = \operatorname{End}_A(T_A)$*

(a) T_A is separating if and only if $\operatorname{pd} X = 1$ for every $X_B \in \mathcal{X}(T_A)$.

(b) T_A is splitting if and only if $\operatorname{id} N = 1$ for every $N_A \in \mathcal{F}(T_A)$.

We have immediately the following corollary.

Corollary 4.1.15 *If A is hereditary, then every tilting module T_A is splitting. If additionally B is hereditary, then T_A is separating.*

We finish this section with a very useful proposition that gives us a relation between Ext-spaces of $\operatorname{mod} A$ and $\operatorname{mod} B$.

Proposition 4.1.16 *Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End}_A(T_A)$. If $M \in \mathcal{T}(T_A)$ and $N \in \mathcal{F}(T_A)$, then, for any $j \geq 1$, there is an isomorphism*

$$\operatorname{Ext}_A^j(M, N) \cong \operatorname{Ext}_B^{j-1}(\operatorname{Hom}_A(T, M), \operatorname{Ext}_A^1(T, N)).$$

In particular if A is hereditary, we have

$$\operatorname{Ext}_A^1(M, N) \cong \operatorname{Hom}_B(\operatorname{Hom}_A(T, M), \operatorname{Ext}_A^1(T, N)).$$

4.2 Exact abelian extension closed subcategories for tilted algebras

As we already discussed in the previous chapters, an exact abelian subcategory is thick if and only if it is closed under extensions and also a thick subcategory is exact abelian if and only if it is closed under arbitrary kernels.

In this section, we shall use another characterisation of these two types of categories. The first proposition is from [Hov], but for completeness we write the proof here. We always assume that the subcategories we are considering are full additive and closed under direct summands.

Proposition 4.2.1 *A full additive subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is exact abelian extension closed if and only if for every exact sequence*

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

the object M_3 is in $\mathcal{C}$ if the objects M_1, M_2, M_4, M_5 are in $\mathcal{C}$.

Proof: Let $\mathcal{C} \subset \mathcal{A}$ be exact abelian subcategory and

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

be exact in $\mathcal{A}$. If $M_1 = M_5 = 0$ and M_2 and M_4 are in $\mathcal{C}$, then M_3 is in $\mathcal{C}$ since $\mathcal{C}$ is exact abelian. Therefore $\mathcal{C}$ is closed under extensions. If $M_1 = M_2 = 0$ and $M_4 = M_5 = 0$, then $\mathcal{C}$ is closed under kernels and cokernels.

Conversely, let $\mathcal{C} \subset \mathcal{A}$ be closed under extensions, kernels and cokernels and let

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

be exact with $M_1, M_2, M_4, M_5 \in \mathcal{C}$. Then $C = \mathrm{Coker}(M_1 \to M_2)$ and $K = \mathrm{Ker}(M_4 \to M_5)$ are in $\mathcal{C}$. We obtain the following diagram:

$$\begin{tikzcd}
M_1 \arrow[r] & M_2 \arrow[rr] \arrow[dr] & & M_3 \arrow[rr] \arrow[dr] & & M_4 \arrow[r] & M_5 \\
& & C \arrow[ur] \arrow[dr] & & K \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

Therefore M_3 is an extension of C and K and hence it is in $\mathcal{C}$. □

Immediately from the definition of a thick category, we get the following proposition.

Proposition 4.2.2 *A full additive subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is thick if and only if for every short exact sequence*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathcal{A}$, if any two of its (non-zero) terms are in $\mathcal{C}$, then the third one is also in $\mathcal{C}$.

From now on, we assume that A is a finite dimensional hereditary k-algebra and we denote by $\operatorname{mod} A$ the category of finite dimensional A-modules. Let T_A be a basic tilting module, $B = \operatorname{End}_A(T_A)$ be its tilted algebra and $(\mathcal{T}(T_A), \mathcal{F}(T_A))$, $\mathcal{X}(T_A), \mathcal{Y}(T_A)$ be the induced torsion pairs in $\operatorname{mod} A$ and $\operatorname{mod} B$, respectively. Since $\operatorname{gl.dim} A \leq 1$, by corollary 4.1.15, the torsion pair in $\operatorname{mod} B$ is splitting. We set $F = \operatorname{Hom}_A(T, -)$, $F' = \operatorname{Ext}^1_A(T, -)$, $G = - \otimes_B T$ and $G' = \operatorname{Tor}^B_1(-, T)$.

Definition 4.2.3 A full additive subcategory $\mathcal{C}$ in $\operatorname{mod} A$ is called **torsion closed** if $M \in \mathcal{C}$ implies $tM \in \mathcal{C}$.

Note that if $\mathcal{C}$ is a thick (or an exact abelian) category, which is torsion closed, then for any object $M \in \mathcal{C}$ we have $M/tM \in \mathcal{C}$.

In this section, we show that there is a bijection between exact abelian extension and torsion closed subcategories in $\operatorname{mod} A$ and exact abelian extension closed subcategories in $\operatorname{mod} B$. In two separate lemmas, we prove each of the directions in the bijection. We denote as before $\operatorname{Thick}(S)$ to be the smallest thick subcategory that contains S, where S is a set of modules. Also if $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, $\mathcal{C} \subseteq \mathcal{A}$ a full subcategory and F a functor from $\mathcal{A}$ to $\mathcal{B}$, then set $F(\mathcal{C})$ to be the full subcategory of $\mathcal{B}$ consisting of objects isomorphic to $F(C)$, for $C \in \mathcal{C}$.

Lemma 4.2.4 *Let $\mathcal{C}$ be an exact abelian extension and torsion closed subcategory in $\operatorname{mod} A$. Then the full subcategory*

$$\mathcal{M} = \{M \in \operatorname{mod} B \mid M = M' \oplus M'', M' \in \operatorname{Hom}_A(T, \mathcal{C}), M'' \in \operatorname{Ext}^1_A(T, \mathcal{C})\}$$

in $\operatorname{mod} B$ is exact abelian and extension closed.

Proof: We divide the proof into two steps. First, we show that $\mathcal{M}$ is thick, and then we show that it is closed under arbitrary kernels. We comment that by definition, $\mathcal{M}$ is closed under direct summands.

Step 1. $\mathcal{M}$ is thick subcategory in $\operatorname{mod} B$. The torsion pair in $\operatorname{mod} B$ is splitting, hence any indecomposable object is either torsion or torsion-free. Take an arbitrary short exact sequence in $\operatorname{mod} B$:

$$0 \to Z_1 \to Z_2 \to Z_3 \to 0,$$

where $Z_i = X_i \oplus Y_i$, $X_i \in \mathcal{M} \cap \mathcal{X}(T_A)$, $Y_i \in \mathcal{M} \cap \mathcal{Y}(T_A)$. We show that if any two of its terms are in $\mathcal{M}$, then the third one is also in $\mathcal{M}$, and then by proposition 4.2.2, the claim shall follow. We apply the functor $G = - \otimes_B T$ to the above sequence, and get the following exact sequence in $\operatorname{mod} A$:

$$0 \to G(X_1) \to G(X_2) \to G(X_3) \to G'(Y_1) \xrightarrow{f} G'(Y_2) \xrightarrow{g} G'(Y_3) \to 0.$$

If, say Z_1, Z_2 are in $\mathcal{M}$, then $G(X_1), G(X_2), G'(Y_1), G'(Y_2)$ are in $\mathcal{C}$ and since the latter is exact abelian extension closed, from proposition 4.2.1 we get the exact sequence

$$G(X_1) \to G(X_2) \to G(X_3) \to G'(Y_1) \to G'(Y_2),$$

and we conclude that $G(X_3) \in \mathcal{C}$. Having in mind that $\mathcal{C}$ is closed under kernels and images, then $\operatorname{Ker} f$ and $G'(Y_2)/\operatorname{Ker} f \cong G'(Y_3)$ are in $\mathcal{C}$. We conclude that Z_3 is in $\mathcal{M}$. The other cases are treated in the same way. This gives an argument for $\mathcal{M}$ to be thick.

Step 2. We prove that if Z_1, Z_2 are arbitrary objects in $\mathcal{M}$ and $f : Z_1 \to Z_2$ a non-zero morphism between them, then $\operatorname{Ker} f$ is in $\mathcal{M}$. We show that we can reduce the proof to one the following cases:

Case 1. $\operatorname{Ker} f \in \mathcal{M}$, where $f : X_1 \to X_2$ and $X_1, X_2 \in \mathcal{M}$ are torsion objects.

Case 2. $\operatorname{Ker} f \in \mathcal{M}$, where $f : Y_1 \to Y_2$ and $Y_1, Y_2 \in \mathcal{M}$ are torsion-free objects.

Case 3. $\operatorname{Ker} f \in \mathcal{M}$, where $f : Y_1 \to X_1$ and $Y_1, X_1 \in \mathcal{M}$ are torsion-free and torsion objects.

To see that, we use a similar argument as in proposition 3.2.12 of the previous chapter. We use induction on $d = \dim(Z_1) + \dim(Z_2)$, where the dimension is over k. Clearly, the assertion holds when $d = 1$. Now, assume that the assertion is true for any morphism $f'\colon Z' \to Z''$ with $\dim(Z') + \dim(Z'') < d$, $Z', Z'' \in \mathcal{M}$. Since $\operatorname{mod} B$ is splitting, we write $Z_i = X_i \oplus Y_i$ $(i = 1, 2)$ with $X_i \in \mathcal{X}(T_A)$ and $Y_i \in \mathcal{Y}(T_A)$ non-zero and $X_1 \oplus Y_1 \xrightarrow{f} X_2 \oplus Y_2$, where $f = \begin{pmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{pmatrix}$ $(f_{12} = \operatorname{Hom}_B(X_1, Y_2) = 0)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "{(1,0)}"] \arrow[d, "f_{11}"] & X_1 \oplus Y_1 \arrow[r, "{\binom{0}{1}}"] \arrow[d, "f"] & Y_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_2 \arrow[r, "\mathrm{Id}"] & Z_2 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Applying the snake lemma(see A.4), we get the exact sequence

$$0 \to \operatorname{Ker} f_{11} \to \operatorname{Ker} f \to Y_1 \to \operatorname{Coker} f_{11} \to \operatorname{Coker} f \to 0.$$

From $\dim(X_1) < \dim(Z_1)$, follows that $\operatorname{Coker} f_{11} \in \operatorname{Thick}(X_1, Z_2) \subseteq \operatorname{Thick}(Z_1, Z_2)$. Since $\operatorname{Coker} f = \operatorname{Coker}(Y_1 \to \operatorname{Coker} f_{11})$, combined with $\dim(Y_1) < \dim(Z_1)$ and

$\dim(\operatorname{Coker} f_{11}) \leq \dim(Z_2)$, we get $\operatorname{Coker} f \in \operatorname{Thick}(Y_1, \operatorname{Coker} f_{11}) \subseteq \operatorname{Thick}(Z_1, Z_2)$, and hence $\operatorname{Ker} f \in \operatorname{Thick}(Z_1, Z_2)$. Since $\operatorname{Coker} f_{11} \in \operatorname{Thick}(Z_1, Z_2) \Leftrightarrow \operatorname{Ker} f_{11} \in \operatorname{Thick}(Z_1, Z_2)$, by the inductional hypothesis we have that $\operatorname{Ker} f_{11} \in \mathcal{M}$ implies $\operatorname{Ker} f \in \mathcal{M}$. Having in mind that $f_{11} : X_1 \to X_2$ is a morphism between torsion-free modules, we land to Case 1.

Now, if happens that $Z_1 = Y_1$ is a torsion-free module, then we consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & Y_1 \arrow[r, "\mathrm{Id}"] \arrow[d, "{(f_{21}, f_{22})}"] & Y_1 \arrow[r] \arrow[d, "f_{22}"] & 0 . \\
0 \arrow[r] & X_2 \arrow[r, "{(1,0)}"] & X_2 \oplus Y_2 \arrow[r, "\binom{0}{1}"] & Y_2 \arrow[r] & 0
\end{tikzcd}$$

Then, as we did above, we apply the snake lemma, and by induction we conclude that $\operatorname{Coker} f_{22} \in \mathcal{M} (\Leftrightarrow \operatorname{Ker} f_{22} \in \mathcal{M})$ implies $\operatorname{Coker} f \in \mathcal{M} (\Leftrightarrow \operatorname{Ker} f \in \mathcal{M})$. Then we land to Case 2, since $f_{22} : Y_1 \to Y_2$ is a morphism between torsion-free modules.

The last possible case is when $Z_1 = Y_1$ is a torsion-free module and $Z_2 = X_1$ is a torsion module.

Case 1. Let Z_1, Z_2 are arbitrary torsion objects in $\mathcal{M}$. For convenience, we write $Z_1 = X_1$ and $Z_2 = X_2$. From the definition of $\mathcal{M}$ follows, that $\mathcal{M} \cap \mathcal{X}(T_A) = \operatorname{Ext}^1_A(T, \mathcal{C}) = \operatorname{Ext}^1_A(T, \mathcal{C} \cap \mathcal{F}(T_A))$. Then an arbitrary morphism $0 \neq f : X_1 \to X_2$ is of the form

$$X_1 = \operatorname{Ext}^1_A(T, C_1) \overset{f = \operatorname{Ext}^1(T, f')}{\longrightarrow} \operatorname{Ext}^1_A(T, C_2) = X_2,$$

where $f' : C_1 \to C_2$, and C_1, C_2 are in $\mathcal{C} \cap \mathcal{F}(T_A)$. We show that $\operatorname{Ker} f \in \mathcal{M}$.

Consider the following diagram in $\operatorname{mod} A$:

$$\begin{tikzcd}
 & C_1 \arrow[rr, "f' = i' \circ \pi'"] \arrow[dr, two heads, "\pi'"] & & C_2 \arrow[dr, two heads] & \\
\operatorname{Ker} f' \arrow[ur, hook] & & \operatorname{Im} f' \arrow[ur, hook, "i'"] & & \operatorname{Coker} f'.
\end{tikzcd}$$

Since $\mathcal{C}$ is an abelian subcategory, we have that $\operatorname{Ker} f', \operatorname{Im} f'$ and $\operatorname{Coker} f'$ are in $\mathcal{C}$, and hence $\operatorname{Ext}^1_A(T, \operatorname{Ker} f')$, $\operatorname{Ext}^1_A(T, \operatorname{Im} f')$,$\operatorname{Ext}^1_A(T, \operatorname{Coker} f')$ and $\operatorname{Hom}_A(T, \operatorname{Coker} f')$ are in $\mathcal{M}$. Moreover, since C_1, C_2 are torsion-free objects in $\operatorname{mod} A$, then $\operatorname{Ker} f' \leq C_1$ and $\operatorname{Im} f' \leq C_2$ are also torsion-free. Now, we apply the functor $F = \operatorname{Hom}_A(T, -)$ to the two short exact sequences above

$$0 \to \operatorname{Im} f' \overset{i'}{\to} C_2 \to \operatorname{Coker} f' \to 0 \qquad (\operatorname{mod} A)$$

$$0 \to F(\operatorname{Coker} f') \to F'(\operatorname{Im} f') \overset{F'(i')}{\to} F'(C_2) \to F'(\operatorname{Coker} f') \to 0 \qquad (\operatorname{mod} B)$$

$$0 \to \operatorname{Ker} f' \to C_1 \overset{\pi'}{\to} \operatorname{Im} f' \to 0 \qquad (\operatorname{mod} A)$$

$$0 \to F'(\operatorname{Ker} f') \to F'(C_1) \overset{F'(\pi')}{\longrightarrow} F'(\operatorname{Im} f') \to 0 \qquad (\operatorname{mod} B)$$

$(F' = \operatorname{Ext}^1_A(T, -))$ and transfer to $\operatorname{mod} B$. We obtain the following diagram:

$$\begin{tikzcd}
X_1 = F'(C_1) \arrow[rr, "f=F'(f')"] \arrow[dr, tail, two heads, "F'(\pi')"] & & F'(C_2) = X_2 \\
& F'(\operatorname{Im} f'). \arrow[ur, "F'(i')"] &
\end{tikzcd}$$

We comment that in general $F'(i')$ is not a monomorphism. Now $f = F'(f') = F'(i' \circ \pi') = F'(i') \circ F'(\pi')$. We have that $\operatorname{Ker} F'(\pi)$ is in $\mathcal{M}$, since $F'(\pi')$ is an epimorphism and $\mathcal{M}$ is closed under kernels of epimorphisms. From the second short exact sequence, we get $\operatorname{Ker} F'(i') \cong F(\operatorname{Coker} f') \in \mathcal{M}$, hence $\operatorname{Ker} F'(i')$ is in $\mathcal{M}$. Applying lemma A.4.2 to $X_1 \overset{F'(\pi')}{\twoheadrightarrow} F'(\operatorname{Im} f') \overset{F'(i')}{\rightarrow} X_2$, we get:

$$0 \to \operatorname{Ker} F'(\pi') \to \operatorname{Ker} f \to \operatorname{Ker} F'(i') \to \operatorname{Coker} F'(\pi') \to \operatorname{Coker} f \to \operatorname{Coker} F'(i') \to 0.$$

Now, since $\operatorname{Coker} F'(\pi') = 0$, and $\mathcal{M}$ is closed under extensions, we have that $\operatorname{Ker} f$ is in $\mathcal{M}$.

Case 2. Let $Z_1, Z_2 \in \mathcal{Y}(T_A)$. The case is analogous to the first case.

Case 3. Let $Z_1 \in \mathcal{X}(T_A)$ and $Z_2 \in \mathcal{Y}(T_A)$. Write $Z_1 = X_1$ and $Z_2 = Y_1$ and denote by $F_1 = G'(X_1)$ and $T_1 = G(Y_1)$. Since A is hereditary, by proposition 4.1.16 we have $\operatorname{Ext}^1_A(T_1, F_1) \cong \operatorname{Hom}_B(Y_1, X_1)$. Let $\eta \in \operatorname{Ext}^1_A(T_1, F_1)$ be the extension that corresponds to the morphism $f : Y_1 \to X_1$. We apply the functor F to η

$$\begin{array}{rcl} \eta : 0 \to F_1 \to M \to T_1 \to 0 & \qquad\qquad & (\operatorname{mod} A) \\ 0 \to F(M) \to Y_1 \xrightarrow{f} X_1 \to F'(M) \to 0 & \qquad\qquad & (\operatorname{mod} B) \end{array}$$

and transfer to $\operatorname{mod} B$. Since $\mathcal{C}$ is closed under extensions and $F_1, T_1 \in \mathcal{C}$, then $M \in \mathcal{C}$ and hence $F(M) \cong \operatorname{Ker} f \in \mathcal{M}$. This finishes the proof in that case as well as of the lemma. □

The next lemma deals with the reverse direction.

Lemma 4.2.5 *Let $\mathcal{M}$ be an exact abelian extension closed subcategory in* $\operatorname{mod} B$. *Then the full subcategory*

$$\mathcal{C} = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) \in \mathcal{M} \textit{ and } \operatorname{Ext}^1_A(T, M) \in \mathcal{M}\}$$

in $\operatorname{mod} A$ *is exact abelian extension and torsion closed.*

Proof: We recall that for an arbitrary module M in $\operatorname{mod} A$, we have $\operatorname{Hom}_A(T, M) = \operatorname{Hom}_A(T, tM)$ and $\operatorname{Ext}^1_A(T, M) = \operatorname{Ext}^1_A(T, M/tM)$, hence we can write

$$\mathcal{C} = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, tM) \in \mathcal{M} \text{ and } \operatorname{Ext}^1_A(T, M/tM) \in \mathcal{M}\}.$$

First, we show that $\mathcal{C}$ is torsion closed. We have that $M \in \mathcal{C}$ if and only if $F(tM) \in \mathcal{M}$ and $F'(M/tM) \in \mathcal{M}$. Now, $t(tM) = tM$ and $tM/t(tM) = 0$, hence $F(t(tM)) = F(tM) \in \mathcal{M}$ and $F'(tM/t(tM)) = 0$, which implies that $tM \in \mathcal{C}$.

Next we show that $\mathcal{C}$ is thick. We notice that by definition $\mathcal{C}$ is closed under direct summands. Now, an arbitrary short exact sequence in $\operatorname{mod} A$ is of the form:

$$0 \to Z_1 \to Z_2 \to Z_3 \to 0,$$

where $Z_i = T_i \oplus R_i \oplus F_i$ $(i = 1, 2, 3)$, $T_i \in \mathcal{T}(T_A)$, $F_i \in \mathcal{F}(T_A)$ and R_i is neither torsion nor torsion-free. We use the functor F and get the following exact sequence in $\operatorname{mod} B$:

$0 \to F(T_1) \oplus F(tR_1) \to F(T_2) \oplus F(tR_2) \to F(T_3) \oplus F(tR_3) \to F'(F_1) \oplus F'(R_1/tR_1) \xrightarrow{f^*} F'(F_2) \oplus F'(R_2/tR_2) \to F'(F_3) \oplus F'(R_3/tR_3) \to 0.$

By proposition 4.2.2, we have three cases to consider, but since they are treated the same, we give the details for only one of the cases, namely assume that $Z_1, Z_2 \in \mathcal{C}$. We show that $Z_3 \in \mathcal{C}$. We have that $F(T_i) \oplus F(tR_i)$ and $F'(F_i) \oplus F'(R_i/tR_i)$ $(i = 1, 2)$ are in $\mathcal{M}$. If we consider the first five non-zero terms of the exact sequence above, then by proposition 4.2.1, we have that $F(T_3) \oplus F(tR_3) \in \mathcal{M}$. Since $\mathcal{M}$ is closed under images and cokernels, we have that $\operatorname{Im} f^* \in \mathcal{M}$, and hence $F'(F_2) \oplus F'(R_2/tR_2)/\operatorname{Im} f^* \cong F'(F_3) \oplus F'(R_3/tR_3)$ is also in $\mathcal{M}$. Now, all of the modules $F(T_3)$, $F(R_3) = F(tR_3)$, $F'(R_3) = F'(R_3/tR_3)$ and $F'(F_3)$ are in $\mathcal{M}$, hence T_3, R_3, F_3 and Z_3 are in $\mathcal{C}$.

To finish the proof, we use theorem 3.3.1. Since $\mathcal{C} \subseteq \operatorname{mod} A$, and A is a hereditary algebra, then $\mathcal{C}$ is exact abelian. □

We are now able to prove the main theorem in this chapter.

Theorem 4.2.6 *Let A be a finite dimensional hereditary k-algebra, T_A a basic tilting module and $B = \operatorname{End}_A(T_A)$. Then the assignments:*

$$\mathcal{C} \overset{i}{\mapsto} \mathcal{M} = \{M \in \operatorname{mod} B \mid M = M' \oplus M'', M' \in \operatorname{Hom}_A(T, \mathcal{C}), M'' \in \operatorname{Ext}^1_A(T, \mathcal{C})\}$$

$$\mathcal{M} \overset{j}{\mapsto} \mathcal{C} = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) \in \mathcal{M} \text{ and } \operatorname{Ext}^1_A(T, M) \in \mathcal{M}\}$$

induce mutually inverse bijections between:

- *exact abelian extension and torsion closed subcategories in* mod A*, and*
- *exact abelian extension closed subcategories in* mod B*.*

Proof: The previous two lemmas showed that the assignments are defined properly. We verify that $(j \circ i)(\mathcal{C}) = \mathcal{C}$, for arbitrary exact abelian extension and torsion closed subcategory $\mathcal{C} \subseteq \operatorname{mod} A$ and $(i \circ j)(\mathcal{M}) = \mathcal{M}$, for arbitrary exact abelian extension closed subcategory $\mathcal{M} \subseteq \operatorname{mod} B$, and the claim shall follow.

(1) $(j \circ i)(\mathcal{C}) = \mathcal{C}$.

"$\supseteq$" Take $C \in \mathcal{C}$ arbitrary. Then $\operatorname{Hom}_A(T, C) \in \operatorname{Hom}_A(T, \mathcal{C}) = \operatorname{Hom}_A(T, \mathcal{C} \cap \mathcal{T}(T_A)) = \mathcal{M} \cap \mathcal{Y}(T_A) \subseteq \mathcal{M} = i(\mathcal{C})$. In the same way $\operatorname{Ext}^1_A(T, C) \in \mathcal{M} = i(\mathcal{C})$. Then $C \in (j \circ i)(\mathcal{C})$.

"$\subseteq$" Take $C \in (j \circ i)(\mathcal{C})$. Then both $\mathrm{Hom}_A(T, C)$ and $\mathrm{Ext}^1_A(T, C)$ are in $i(\mathcal{C}) = \mathcal{M}$. Note that by construction $(j \circ i)(\mathcal{C})$ is torsion closed, hence both tC and C/tC are in $(j \circ i)(\mathcal{C})$. We have that $\mathrm{Hom}_A(T, C) = \mathrm{Hom}_A(T, tC) \in \mathcal{M} \cap \mathcal{Y}(T_A) = \mathrm{Hom}_A(T, \mathcal{C}) = \mathrm{Hom}_A(T, \mathcal{C} \cap \mathcal{T}(T_A))$ and therefore by theorem 4.1.11(b) $tC \in \mathcal{C} \cap \mathcal{T}(T_A)$. In the same way we have that $C/tC \in \mathcal{C} \cap \mathcal{F}(T_A)$. Taking the canonical sequence for C, and having in mind that $\mathcal{C}$ is extension closed, we have that $C \in \mathcal{C}$.

(2) $(i \circ j)(\mathcal{M}) = \mathcal{M}$. Since the torsion pair $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ in $\mathrm{mod}\, B$ is splitting, any $M \in \mathrm{mod}\, B$ is of the form $M = M_1 \oplus M_2$, where M_1 is a torsion-free module and M_2 is a torsion module.

"$\supseteq$" Take an arbitrary object $M \in \mathcal{M}$. Then there are objects $X, Y \in \mathrm{mod}\, A$ such that $\mathrm{Hom}_A(T, tX) = M_1$ and $\mathrm{Ext}^1_A(T, Y/tY) = M_2$. But then $\mathrm{Hom}_A(T, tX) \in \mathrm{Hom}_A(T, j(\mathcal{M})) = \{\mathrm{Hom}_A(T, Z) \mid Z \in \mathrm{mod}\, A,\text{ such that } \mathrm{Hom}_A(T, Z) \in \mathcal{M} \text{ and } \mathrm{Ext}^1_A(T, Z) \in \mathcal{M}\}$, since $\mathrm{Ext}^1_A(T, tX) = 0$, that is, $M_1 \in (i \circ j)(\mathcal{M})$. Similarly, $\mathrm{Ext}^1_A(T, Y/tY) \in \mathrm{Ext}^1_A(T, j(\mathcal{M}))$, that is, $M_2 \in (i \circ j)(\mathcal{M})$. Hence $M \in (i \circ j)(\mathcal{M})$.

"$\subseteq$" Take an arbitrary object M in $(i \circ j)(\mathcal{M}) \subseteq \mathrm{mod}\, B$. Then $M = M_1 \oplus M_2$, where $M_1 = \mathrm{Hom}_A(T, X)$ and $M_2 = \mathrm{Ext}^1_A(T, Y)$, for $X, Y \in j(\mathcal{M}) \subseteq \mathrm{mod}\, A$. By definition of $j(\mathcal{M})$, both $\mathrm{Hom}_A(T, X)$ and $\mathrm{Ext}^1_A(T, Y)$ are in $\mathcal{M}$, and hence M_1 and M_2 are in $\mathcal{M}$. □

Before we give a corollary of the theorem, we need to recall some facts from the theory of quiver representations.

Let $Q = (Q_0, Q_1, s, t)$ be a finite, connected, and acyclic quiver and let $n = |Q_0|$. For every point $a \in Q_0$, we define a new quiver $\sigma_a Q = (Q'_0, Q'_1, s', t')$ as follows: All the arrows of Q having a as a source or as target are reversed, all others arrows remain unchanged. An **admissible sequence of sinks** in a quiver Q is defined to be a total ordering $(a_1, \ldots, a_n)$ of all points in Q such that:

(i) a_1 is a sink in Q, and

(ii) a_i is a sink in $\sigma_{a_{i-1}} \ldots \sigma_{a_1} Q$ for every $2 \leq i \leq n$.

We have the following proposition.

Proposition 4.2.7 *[AS, Chapter VII.5, Proposition 5.2] Let Q and Q' be two trees having the same underlying graph. There exists a sequence $i_1, \ldots, i_t$ of points of Q such that $\sigma_{i_t} \ldots \sigma_{i_1} Q = Q'$.*

Let A be a finite dimensional hereditary k-algebra, which we assume that is not simple. There exists an algebra isomorphism $A \cong kQ_A$, where Q_A is a finite, connected, and acyclic quiver. Then there exists a sink $a \in (Q_A)_0$ that is not a source, so that the simple A-module $S(a)_A$ is projective and non-injective. Consider the following module in $\operatorname{mod} A$

$$T[a]_A = \tau^{-1}S(a) \oplus (\bigoplus_{b \neq a} P(b)).$$

It is not difficult to check that it is a tilting module, see [AS, Chapter VI.2.8], which is called **APR-tilting**.

We may ask whether there a connection between path algebras kQ_A and $k(\sigma_a Q_A)$, where σ_a is a reflection at the sink a for the quiver Q_A. The following proposition gives an answer to that question.

Proposition 4.2.8 *[AS, Chapter VII.5, Proposition 5.3] Let A be a basic hereditary and non-simple algebra, a be a sink in its quiver Q_A, and $T[a]$ be the APR-tilting A-module at a. Then the algebra $B = \operatorname{End} T[a]_A$ is isomorphic to $k(\sigma_a Q_A)$.*

Now, we prove the following proposition.

Proposition 4.2.9 *Let Q be a finite acyclic quiver, a be a sink and $\sigma_a Q$ be the reflected at a quiver Q. Then there is a bijection between exact abelian extension closed subcategories in* $\operatorname{mod} kQ$ *and* $\operatorname{mod} k(\sigma_a Q)$.

Proof: By theorem 4.2.6, we have a bijection between exact abelian extension and torsion closed subcategories in $\operatorname{mod} kQ$ and exact abelian extension closed subcategories in $\operatorname{mod} k(\sigma_a Q)$. But since $k(\sigma_a Q)$ is hereditary, then the APR-tilting module is separating. Then the torsion pair $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ in $\operatorname{mod} kQ$ splits and hence every exact abelian extension closed subcategory in $\operatorname{mod} kQ$ is also torsion closed.□

Corollary 4.2.10 *Let Q and Q' be a finite acyclic quivers, having the same underlying graph but with different orientations. Then there is a bijection between exact abelian extension closed subcategories in* $\operatorname{mod} kQ$ *and* $\operatorname{mod} kQ'$.

Proof: By proposition 4.2.7, we have that there exists a sequence $i_1, \ldots, i_t$ of points of Q such that $\sigma_{i_t} \ldots \sigma_{i_1} Q = Q'$. Set $\sigma_{i_0} Q = Q$. By proposition 4.2.9, we have that for $k = 0, \ldots, t-1$, there is a bijection between exact abelian extension closed subcategories of $\operatorname{mod} k(\sigma_{i_k} \ldots \sigma_{i_0} Q)$ and $\operatorname{mod} k(\sigma_{i_{k+1}} \ldots \sigma_{i_0} Q)$. Since $\operatorname{mod} k(\sigma_{i_t} \ldots \sigma_{i_1} Q) = \operatorname{mod} kQ'$, the proof follows. □

Remark 4.2.11 We point out that Kristian Brüning showed the same result, see [Br2, Corollary 5.6]. There he established a bijection between thick subcategories of bounded derived category $\mathcal{D}^b(\mathcal{A})$ and exact abelian extension closed subcategories in $\mathcal{A}$, where $\mathcal{A}$ is a hereditary abelian category. Then using the fact that $\mathcal{D}^b(\operatorname{mod} kQ) \cong \mathcal{D}^b(\operatorname{mod} kQ')$, see [Ha, Proposition 4.5], the claim follows.

Appendix A

Basic and auxiliary results

A.1 Quivers and their representations

The reference for this section is [AS, Chapter II.1, Chapter III.1].

A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0 (whose elements are called *vertices*) and Q_1 (whose elements are called *arrows*), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$ respectively.

We denote a quiver $Q = (Q_0, Q_1, s, t)$ simply by Q. A *subquiver* of a quiver $Q = (Q_0, Q_1, s, t)$ is a quiver $Q' = (Q'_0, Q'_1, s', t')$ such that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$ and the restrictions $s_{Q'_1}$, $t_{Q'_1}$ of s, t to Q'_1 are respectively equal to s', t'. Such a subquiver is called *full* if Q'_1 equals the set of all those arrows in Q_1 whose source and target both belong to Q'_0.

Example A.1.1 The quiver

$$\begin{tikzcd}\Delta_n : 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n\end{tikzcd}$$

is a subquiver of the quiver

$$\begin{tikzcd}\tilde{\Delta}_n : 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend left] ,\end{tikzcd}$$

and a full subquiver of the quiver

$$\begin{tikzcd}\Delta_{n+1} : 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[r] & n+1 .\end{tikzcd}$$

A quiver Q is said to be **finite** if Q_0 and Q_1 are finite sets. The quiver Q is said to be **connected** if its underlying graph is connected.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $a, b \in Q_0$. A path of length $\ell \geq 1$ with source a and target b is a sequence

$$(a|\alpha_1, \alpha_2, \ldots, \alpha_\ell|b),$$

where $\alpha_k \in Q_1$ for all $1 \leq k \leq \ell$, and we have $s(\alpha_1) = a$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k \leq \ell$, and finally $t(\alpha_\ell) = b$. To each point $a \in Q$ a path of length $\ell = 0$ is called the **trivial path** at a and it is denoted by ϵ_a. Thus, the paths of lengths 0 and 1 are in bijective correspondence with the elements of Q_0 and Q_1, respectively. A path of length $\ell \geq 1$ is called **cycle** whenever its source and target coincide. A quiver is called **acyclic** if it contains no cycles.

Let Q be a quiver. The **path algebra** kQ of Q is the k-algebra whose underlying k-vector space has as its basis the set of all paths $(a|\alpha_1, \alpha_2, \ldots, \alpha_\ell|b)$ of length $\ell \geq 0$ in Q and such that the product of two basis vectors $(a|\alpha_1, \alpha_2, \ldots, \alpha_\ell|b)$ and $(c|\beta_1, \beta_2, \ldots, \beta_k|d)$ of kQ is defined by

$$(a|\alpha_1, \alpha_2, \ldots, \alpha_\ell|b)(c|\beta_1, \beta_2, \ldots, \beta_k|d) = \delta_{bc}(a|\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_k|d),$$

where δ_{bc} is the Kronecker delta:

$$\delta_{bc} = \begin{cases} 0 & \text{if } t(\alpha_\ell) \neq s(\beta_1) \\ 1 & \text{if } t(\alpha_\ell) = s(\beta_1). \end{cases}$$

In other words, the product of two paths $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ and $\beta_1, \beta_2, \ldots, \beta_k$ is equal to zero if $t(\alpha_\ell) \neq s(\beta_1)$ and is equal to the composed path $\alpha_1, \alpha_2, \ldots, \alpha_\ell \beta_1, \beta_2, \ldots, \beta_k$ if $t(\alpha_\ell) = s(\beta_1)$. The product of basis elements is then extended to arbitrary elements of kQ by distributivity.

Lemma A.1.2 *Let Q be a quiver and kQ be its path algebra. Then*

(a) kQ is an associative algebra;

(b) kQ has an identity element if and only if Q_0 is finite;

(c) kQ is finite dimensional if and only if Q is a finite and acyclic.

We point out that for a given path algebra kQ, there is a direct sum decomposition

$$kQ = kQ_0 \oplus kQ_1 \oplus kQ_2 \oplus \cdots \oplus kQ_\ell \ldots$$

of the k-vector space kQ, where, for each $\ell \geq 0$, kQ_ℓ is the subspace of kQ generated by the set Q_ℓ of all paths of length ℓ. It is easy to see that $(kQ_n) \centerdot (kQ_m) \subseteq kQ_{n+m}$ for all $n, m \geq 0$, which shows that kQ is a **graded algebra**.

Definition A.1.3 Let Q be a finite and connected quiver. The two-sided ideal of the path algebra kQ generated (as an ideal) by the arrows of Q is called the **arrow ideal** of kQ and is denoted by R_Q.

Note that there is a direct sum decomposition

$$R_Q = kQ_1 \oplus kQ_2 \oplus \cdots \oplus kQ_\ell \oplus \cdots$$

of the k-vector space R_Q, where kQ_ℓ is the subspace of kQ generated by the set Q_ℓ of all paths of length ℓ. In particular, the underlying k-vector space of R_Q is generated by all paths in Q of length $\ell \geq 1$. This implies that, for each $\ell \geq 1$,

$$R_Q^\ell = \bigoplus_{m \geq \ell} kQ_m$$

and therefore R_Q^ℓ is the ideal of kQ generated, as a k-vector space, by the set of all paths of length $\geq \ell$.

Definition A.1.4 Let Q be a finite quiver. A **representation** M of Q is defined by the following data:

(1) To each point a in Q_0 is associated a k-vector space M_a;

(2) To each arrow $\alpha : a \to b$ in Q_1 is associated a k-linear map $\phi_\alpha : M_a \to M_b$.

Such a representation is denoted as $M = (M_a, \phi_a)_{a \in Q_0, \alpha \in Q_1}$, or simply $M = (M_a, \phi_\alpha)$. It is called **finite dimensional** if each vector space M_a is finite dimensional.

Let $M = (M_a, \phi_\alpha)$ and $M' = (M'_a, \phi'_\alpha)$ be two representations of Q. A **representation morphism** $f : M \to M'$ is a family $f = (f_a)_{a \in Q_0}$ of k-linear maps $(f_a : M_a \to M'_a)_{a \in Q_0}$ that are compatible with the structure maps ϕ_α that is, for each arrow $\alpha : a \to b$, we have $\phi'_a f_a = f_b \phi_\alpha$ or equivalently, the following square is commutative:

$$\begin{tikzcd}
M_a \arrow[r, "\phi_\alpha"] \arrow[d, "f_a"] & M_b \arrow[d, "f_b"] \\
M'_a \arrow[r, "\phi'_\alpha"] & M'_b
\end{tikzcd}$$

Let $f : M \to M'$ and $g : M' \to M''$ be two morphisms of representations of Q, where $f = (f_a)_{a \in Q_0}$ and $g = (g_a)_{a \in Q_0}$. Their composition is defined to be the family $gf = (g_a f_a)_{a \in Q_0}$. Then gf is easily seen to be a morphism from M to M''. We have defined a category $\mathrm{Rep}_k(Q)$ of k-linear representations of Q. We denote by $\mathrm{rep}_k(Q)$ the full subcategory of $\mathrm{rep}(Q)$ consisting of the finite dimensional representations.

Lemma A.1.5 *Let Q be a finite quiver. Then $\mathrm{Rep}_k(Q)$ and $\mathrm{rep}_k(Q)$ are abelian k-categories.*

A.2 Hereditary algebras

In the whole thesis, we considered hereditary algebras and their module categories. Here we recall what a hereditary algebra is, and we point out the connection with the quiver representations. The reference is [AS, Chapter VIII]. In this section k is an algebraically closed field.

Definition A.2.1 Let A be a finite dimensional k-algebra. The following assertions are equivalent:

- A is hereditary;
- Submodules of projective modules are projective;
- The global dimension of A is at most one;
- $\mathrm{Ext}^i_A(M, N) = 0$ for all A-modules M and N and for all $i \geq 2$.

Example A.2.2 If Q is a finite, connected, and acyclic quiver, then the algebra $A = kQ$ is hereditary.

The next theorem relates modules and representations.

Theorem A.2.3 *Let A be a basic and connected finite dimensional hereditary k-algebra. There exists a finite and acyclic quiver Q_A such that*

$$\mathrm{Mod}\, A \xrightarrow{\cong} \mathrm{Rep}_k(Q)$$

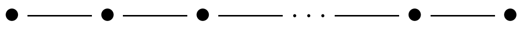
is k-linear equivalence of categories, that restricts to an equivalence of categories $\mathrm{mod}\, A \xrightarrow{\cong} \mathrm{rep}_k(Q)$.

Definition A.2.4 A finite dimensional k-algebra A is said to be **representation-finite** (or an algebra of **finite representation type**) if the number of the isomorphism classes of indecomposable finite dimensional right A-modules is finite. A k-algebra A is called **representation-infinite** (or an algebra of **infinite representation type**) if A is not representation-finite.

By a result of Gabriel, representation-finite hereditary algebras are classified.

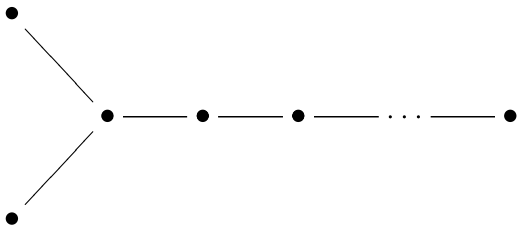
Theorem A.2.5 (Gabriel) *Let Q be a finite, connected, and acyclic quiver; k be an algebraically closed field; and $A = kQ$ be the path k-algebra of Q. The algebra A is representation finite if and only if the underlying graph $\overline{Q}$ of Q is one of the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, with $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$.*

$\mathbb{A}_n$:



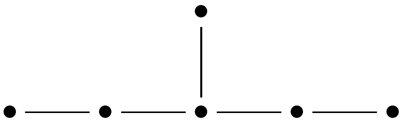
$(n \geq 1)$

$\mathbb{D}_n$:

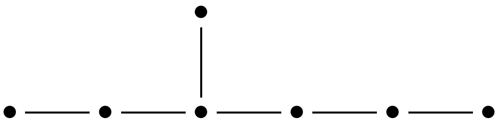


$(n \geq 4)$

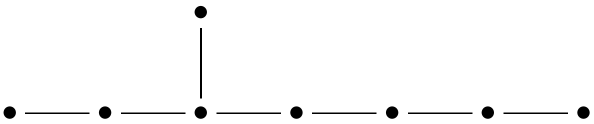
$\mathbb{E}_6$:



$\mathbb{E}_7$:



$\mathbb{E}_8$:



A.3 The Auslander-Reiten quiver

In this section, we turn to the structure theory of the module category. Fix a finite dimensional hereditary k-algebra A. There is a special quiver, called **Auslander-Reiten** quiver, that combinatorially encodes the building blocks of $\operatorname{mod} A$, namely the indecomposable modules and the irreducible morphisms.

First, we recall he following fundamental theorem, that reduces the study of modules to indecomposable modules.

Theorem A.3.1 (Krull-Schmidt) *Let A be a finite dimensional k-algebra. For a finitely generated A-module M there are indecomposable A-modules $M_1, \ldots, M_n$ such that $M \cong \oplus_{i=1}^{n} M_i$. Furthermore, the modules $M_1, \ldots, M_n$ are unique up to permutation.*

Definition A.3.2 Let A be a finite dimensional k-algebra. A morphism of A-modules $f : M \to N$ is an **irreducible morphism**, if

(i) f is neither a section nor a retraction, and

(ii) if $f = f_1 \circ f_2$, then either f_1 is a retraction or f_2 is a section.

Denote by $\operatorname{Irr}(M, N)$ the k-vector space of irreducible morphisms from M to N. As for objects the study of morphisms is reduced to the study of irreducible ones.

Theorem A.3.3 *Let A be a finite dimensional k-algebra of a finite representation type. Every morphism between finitely presented indecomposable A-modules that is not invertible is a finite sum of finite compositions of irreducible maps.*

Definition A.3.4 The Auslander-Reiten quiver (AR-quiver) $\Gamma(A)$ of the algebra A has as vertices the isomorphism classes of indecomposable modules. The arrows from $[M]$ to $[N]$ correspond bijectively to a k-basis of the vector space of irreducible maps $\mathrm{Irr}(M,N)$. The quiver $\Gamma(A)$ is locally finite in the sense that every vertex has only finitely many neighbors. The Auslander-Reiten quiver is equipped with an extra structure: the translate. It is a bijective map

$$\tau : \Gamma(A) \setminus \mathrm{Proj}(A) \to \Gamma(A) \setminus \mathrm{Inj}(A),$$

where $\mathrm{Proj}(A)$ and $\mathrm{Inj}(A)$ denote the sets of isomorphism classes of indecomposable projective and injective modules, respectively.

The following notion is central for the structure of the AR-quiver.

Definition A.3.5 A short exact sequence

$$0 \to L \to M \to N \to 0$$

is called **almost split** or an **Auslander-Reiten sequence (AR-sequence)**, if L and N are indecomposable and the maps $L \to M$ and $M \to N$ are irreducible. The following theorem describes the relation between an indecomposable module N and its translate τN.

Theorem A.3.6 *Let A be a finite dimensional algebra over an algebraically closed field k. For every indecomposable non-projective A-module N there is an AR-sequence*

$$0 \to \tau N \to \bigoplus_{i=1}^{n} M_i^{n_i} \to N \to 0$$

in which $n_i \geq 0$ and the modules M_i are pairwise non-isomorphic indecomposable. Furthermore, $n_i = \dim_k \mathrm{Irr}(M_i, N) = \dim_k \mathrm{Irr}(\tau N, M_i)$.

We finish this section with the following important formula that expresses the translate homologically.

Theorem A.3.7 (Auslander-Reiten formulas) *Let A be a hereditary algebra, and M, N be A-modules. There exist functorial isomorphisms*

$$\tau M \cong D\,\mathrm{Ext}_A^1(M,A) \text{ and } \tau^{-1}M \cong \mathrm{Ext}_A^1(DM,A).$$

Moreover,

$$\mathrm{Ext}_A^1(M,N) \cong D\,\mathrm{Hom}_A(N,\tau M) \text{ and } \mathrm{Ext}_A^1(M,N) \cong D\,\mathrm{Hom}_A(\tau^{-1}N,M).$$

We comment that these formulas are valid in a larger context. We refer to [Kr] for a very elegant proof of these formulas.

A.4 Two homological facts

In the last section, we recall two standard facts from the homological algebra. In this section R is an associative ring.

Lemma A.4.1 (Snake Lemma) *Consider a commutative diagram of R-modules of the form*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C'. & \end{tikzcd}$$

If the rows are exact, there is an exact sequence

$$\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \to \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h.$$

Moreover, if $A \to B$ is a monomorphism, then so is $\operatorname{Ker} f \to \operatorname{Ker} g$, and if $B' \to C'$ is an epimorphism, then so is $\operatorname{Coker} g \to \operatorname{Coker} h$.

The proof can be found in [Wb, Chapter 1].

Corollary A.4.2 *If we have maps $A \xrightarrow{\psi} B \xrightarrow{\phi} C$ of R-modules, then there is an exact sequence*

$$0 \to \operatorname{Ker} \psi \to \operatorname{Ker} \phi\psi \to \operatorname{Ker} \phi \to \operatorname{Coker} \psi \to \operatorname{Coker} \phi\psi \to \operatorname{Coker} \phi \to 0.$$

Proof: Applying the snake lemma to the following commutative diagram

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & A \arrow[r, "\mathrm{Id}"] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, "\phi\psi"] & 0 \\ 0 \arrow[r] & \operatorname{Ker} \phi \arrow[r] & B \arrow[r, "\phi"] & \operatorname{Im} \phi, & \end{tikzcd}$$

we get $0 \to \operatorname{Ker} \psi \to \operatorname{Ker} \phi\psi \to \operatorname{Ker} \phi \to \operatorname{Coker} \psi \to \operatorname{Im} \phi / \operatorname{Im} \phi\psi \to 0$, since $B \to \operatorname{Im} \phi$ is an epimorphism. Then using the third isomorphism theorem for the modules $\operatorname{Im} \phi\psi \leq \operatorname{Im} \phi \leq C$, we obtain $0 \to \operatorname{Im} \phi / \operatorname{Im} \phi\psi \to C / \operatorname{Im} \phi\psi \to C / \operatorname{Im} \phi \to 0$, and gluing with the above sequence, the claim follows. □

Proposition A.4.3 *[ARS, Chapter 1, Proposition 2.6] Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] & B \arrow[d, "g"] \\ B' \arrow[r, "g'"'] & C \end{tikzcd}$$

be a commutative diagram of morphisms between R-modules.

(a) The following are equivalent

- *(i) The diagram is a push-out diagram;*
- *(ii) The induced sequence* $A \overset{\binom{f}{-f'}}{\to} B \coprod B' \overset{(g,g')}{\to} C \to 0$ *is exact;*
- *(iii) In the induced exact commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"] & B \arrow[r] \arrow[d, "g"] & \operatorname{Coker} f \arrow[r] \arrow[d, "h"] & 0 \\
B' \arrow[r, "g'"] & C \arrow[r] & \operatorname{Coker} g' \arrow[r] & 0
\end{tikzcd}$$

  h *is an isomorphism.*

(b) The following are equivalent

- *(i) The diagram is a pull-back diagram;*
- *(ii) The induced sequence* $0 \to A \overset{\binom{f}{-f'}}{\to} B \coprod B' \overset{(g,g')}{\to} C$ *is exact;*
- *(iii) In the induced exact commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} f \arrow[r] \arrow[d, "h"] & A \arrow[r, "f"] \arrow[d, "f'"] & B \arrow[d, "g"] \\
0 \arrow[r] & \operatorname{Ker} g' \arrow[r] & B' \arrow[r, "g'"] & C
\end{tikzcd}$$

  h *is an isomorphism.*

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